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ABSTRACT

Mathematical skills and reasoning and their applications are considered as forming one of the basic and essential areas of knowledge and experience that should be available to all Australian children. This collection of essays explaining problem solving in mathematics as an organizing theme for current and future curriculum development is a means of informing people of the centrality of the theme and of beginning the essential task of teacher development. Essays consider real problem solving and a problem-oriented curriculum approach; how habituated behaviors and memory contribute to the development of heuristic strategies; using developmental efforts to emphasize problem solving; criteria for real-life problems; important variables in increasing school emphasis on mathematical modeling; and selecting content based on adult needs and immediate interests. Finally, a position paper on problem solving is given. (MNS)

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Applying Problem Solving

Readings in Mathematics Curriculum and Instruction for Australian Schools.

Australian Mathematics Education Program.

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Curriculum Development Centre
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APPLYING PROBLEM SOLVING:

READINGS IN MATHEMATICS CURRICULUM AND INSTRUCTION FOR AUSTRALIAN SCHOOLS

Australian Mathematics Education Program

AMEP is a Curriculum Development Centre co-operative curriculum development project based in the University of Queensland

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Foreword

The overall aims of the Australian Mathematics Education Program are explained in depth in the last paper of this collection, but in simple terms they are concerned with building teacher confidence and competence and improving student attitudes and achievement through emphasis on meaning in mathematics and relevance to real life situations.

Thus the Australian Mathematics Education Program sees problem solving as a fundamental and central activity in the teaching and learning of mathematics.

Such a view is consistent with, and in fact, underscores the CDC's belief that the challenge for schooling in the eighties and ahead is to provide all students with learning tasks that are well-structured, flexible, socially relevant, future-oriented and stimulating.

The Centre has nominated mathematical skills and reasoning and their applications as one of the basic and essential areas of knowledge and experience that should be available to all Australian children (Core Curriculum for Australian Schools). Knowledge alone does not equate with basic and essential learnings: there is a rich variety of learning processes and learning environments that are likewise integral.

In publishing this collection of essays explaining problem solving in mathematics as an organising theme for current and future curriculum development, the CDC is wishing to inform people of the centrality of the themes to AMEP and to begin the essential task of teacher development. I have pleasure in commending this volume for your study.

David Francis
Director

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Editors' Introduction

The first six essays of this volume were prepared for an April 1981 meeting of the Steering Committee of the Australian Mathematics Education Program (AMEP). Prior to that meeting, appropriate program goals had been set as helping teachers work in problem solving and its applications, and helping students become more confident and competent in mathematics. The April meeting used these papers to stimulate ideas and discussion about what steps AMEP should undertake in developing program activities.

The first three essays deal with problem solving. Burkhardt details stylistic teaching variables and canvasses what is at stake if we do not help students and teachers broaden their view of the nature of problem solving. He advocates a problem oriented curriculum approach. Osborne suggests elements of traditional curriculum concern – habituated behaviours and memory – contribute to the acquisition of heuristic strategies. Henry argues the need to use developmental efforts as a way to emphasise more about the learning and teaching of problem solving. Each of the authors brings a particular insight into the uses of real world situations as the context for problem solving.

Applying mathematics is the focus of the remaining three essays. Ormell describes criteria for real life problems 'with bite' that can motivate learners. Treilibs and Gaffney narrow the domain for processes in applying mathematics by examining some important variables in any increase of the school emphasis on mathematical modelling. Edwards advocates relating content selection to the mathematical skills needed by the 'average' adult, and paralleling these skills with analogous problems that have meaning at the learner's stage of development. Each of the authors gives attention to problem solving processes through an applications orientation.

We believe, with the authors, that problem solving and applying mathematics are inextricably linked. We also believe that increasing the emphases on these activities in the classroom is a significant challenge for all mathematics educators. To help develop classroom attention to problem solving, a planning guide for developing a lesson around problem solving is provided. The idea for the structured planning of lessons on problem solving was suggested by materials prepared for *The Skills and Procedures of Mathematical Problem Solving Project* (SSRC 1980). Paralleling this planning guide, a comparable guide for working in mathematical modelling is given.

The first page of each guide is an inventory of organisers, procedures and skills suitable for problem solving or modelling. Organisers are essentially 'start up' methods for use early in the lesson. Procedures are thinking strategies most useful after the situation is understood or begun. The second page has at the centre the problem statement or, for modelling, the situation description, and

the planning variables specified around the border. Inputs needed to be considered for the lesson and the outputs or objectives are indicated in the right hand column and keyed to the inventory.

The third page indicates a structure for one possible solution (resolution and notes section), suggests key interventions the teacher might make if students encounter difficulties and, finally, indicates connections and extensions to other mathematical topics that are possible and/or interesting.

A cautionary note is given about the adaptation of the problem solving inventory to modelling. Mathematics educators have been concerned with teaching modelling for only a short while. Whereas the history of research effort and the practical traditions of teaching provide guidance for the design of the problem solving inventory, no comparable experience has helped design the inventory for modelling. Indeed, as you work with model building activities, you should consider the inventory as suggestions to try, in order to test their usefulness as a construct that can support instruction.

The seventh and final essay is a position paper on problem solving and applications that sets out the beliefs of the AMEP Standing Committee. It presents their rationale for electing to use problem solving and applications of mathematics as a vehicle for change in curriculum and instruction in mathematics in Australian schools.

Although the essay was drafted by Marjorie Carss and Alan Osborne, it was written to an outline negotiated and argued by the Standing Committee. As such, it represents a statement of commitment by leaders in mathematics education from all the states and territories on desired changes in mathematics experiences for Australian children and youth.

MARJORIE CARSS
Director, AMEP

ALAN OSBORNE
Consultant, AMEP

The Role of AMEP

By adopting a multi-faceted approach to problem solving (including applications), the Australian Mathematics Education Program will provide an opportunity for many professional mathematics educators (teachers, lecturers, researchers) to contribute their insights, experience and wisdom so that AMEP can capitalise on the rich store of talent in mathematics education in Australia. Consequently, AMEP plans to devote the major portion of its resources to enable mathematics educators to test and evaluate their ideas and to encourage and enhance the communication of productive approaches to teaching problem solving and the application of mathematics. We believe that experimentation and development happens most fruitfully at the classroom and school level where instructional and curriculum principles are put into practice. Activities are often enriched by being co-ordinated in a number of schools with shared goals.

AMEP is calling for proposals from mathematics educators at all levels for projects in three major categories:

A. **Development Projects**

Projects in this category would include those that seek to identify and develop teaching strategies, instructional ideas and classroom materials that enhance the teaching of aspects of problem solving. Materials developed would need to be tested for their effectiveness in reaching stated goals, and their impact on learners and teachers ascertained. The criteria for funding projects assign a priority to proposals that seek to trial and test new ventures in curriculum or instruction.

B. **Adaptation Projects**

Projects in this category would include those that focus on the implementation within a local system of curricular and/or instructional approaches developed in other schools or institutions. The criteria for funding such projects places a premium on adjusting and modifying the materials or approaches to local contexts. It should be noted that a large part of the rationale for this kind of project is to encourage the spread of ideas and information about productive practices, about teaching and learning problem solving between schools, regions and states.

C. **Dissemination Projects**

Projects in this category would include those that focus on the communication of ideas and materials in easily usable formats and by processes that assure a wide impact across many schools. In-service structures that encourage the sharing of ideas and experiences would come under this category.

The overall intent of AMEP is to facilitate and encourage the development and spread of sound practices in the teaching and learning of aspects of problem solving in mathematics and the incorporation of these practices into the total mathematics curriculum.

Real Problem Solving and the Mathematics Curriculum

Hugh Burkhardt

Mathematics and the Present Curriculum

Mathematics is an active subject – its pure intellectual challenges are balanced by its power in helping us to tackle problems from outside mathematics. The main tasks we ask our students to achieve in either area are all problem solving of various kinds. The amount of factual information to be learnt in mathematics at any given level is probably less than in any other subject while the techniques in which we would like the mastery that makes them routine are relatively few in number. So we might expect the amount of time we spend on routine recall or exercise learning to be quite small. Why then is this AMEP initiative so unusual? Indeed, why is it needed?

The answer lies mainly in the yawning gulf between aim and achievement in the school mathematics experience of most children. My direct experience relates mainly to English and American schools but I believe that much of what I say is paralleled in Australia and elsewhere. Targets are set which are wholly unrealistic when compared with the full set of measures of understanding which we may describe as:

1. imitative – of a technique just demonstrated or a closely similar example (near transfer), or the recall of a fact;
2. retention – doing the same after several months;
3. explanation – describing what you are doing and why;
4. extension – of the same techniques to a less obviously similar case (farther transfer);
5. adaptation – of the technique itself for other purposes.

The school curriculum stresses 1 with some of 2 and, in the UK, awards passes at a low level of competence (≃ 40% in examinations). Tasks are chosen, therefore, which 'stretch' the pupils involved. Some success in imitation is achieved, but the level of retention is so low that the same material must be tackled again and again; the 'spiral curriculum' is in fact circular for most children.

> We teach them fractions at 11, 12, 13 and 14. They still don't understand. What shall we do at 15?

That was a teacher's question at a Shell Centre seminar. The answer in practice is usually practice; more fractions. Thus the balance of mathematics itself and of its applications, emphasising problem solving which involves extension and

adaptation in an essential way, is destroyed. Because the targets for the mathematical tool-kit of learnt techniques are never reached, the use of the tool-kit hardly enters.

I have avoided the word 'objectives', since these are supposed by definition to be reached. I believe that few people would be willing to write down 'the imitative exercising of the basic techniques of arithmetic, and a few others, without reliable accuracy' as the objectives of a secondary mathematics curriculum. The aims normally written down as objectives are infinitely more ambitious; their connection to reality really is as loose as I described.'

The purpose of the gloomy analysis is twofold. This critique of the present situation suggests that there is plenty of room for improvement and reason for change. However, it is equally important not to *assume* that the ideas for an improved curriculum that we adopt will be more successful. This must be demonstrated not just on a pilot scale but with the full range of pupils and, more importantly, of teachers for whom it is designed. The present curriculum represents in a very real sense our best efforts so far. It is not for lack of discussion or experiment that mathematical education is as ineffective as it is.

It is true that remarkable and inspiring successes have been achieved by enterprising teachers and projects in various ways and at various times and places but the assumption that all, or even most teachers can achieve comparable results is dubious at the very least. Mixed ability teaching (by teachers) as well as mixed ability learning (by students) is a fact of life.

Teaching Styles and Their Different Challenges

Let me give an example. In England it is widely believed that problem solving skills are best developed by investigative methods of teaching in which students take a large share of responsibility for the strategies they use in tackling problems. There is evidence to support this view and the ATM, with nearly 10 000 members among the 30 000 secondary and 200 000 primary teachers of mathematics, has for two decades encouraged teachers to develop the skills necessary for this approach. Nonetheless, the recent survey by HM Inspectors, who looked at over 55 000 hours of secondary teaching, failed to find any significant amount of such investigations in mathematics lessons. I think there are three ways in which such teaching puts greater demands on the teacher, which go a fair way to explaining this:

1. Mathematically – the teacher must follow the student's line of reasoning, perceive its soundness and its likely outcomes, diagnose any difficulties that are likely to arise and provide minimum correction (not just re-state the 'standard' line of argument the teacher has prepared).
2. Pedagogically – the teacher must handle a much more varied set of learning situations, with each student likely to be at a different place with different difficulties, and be much more sensitive in providing help.
3. Personally – the teacher must accept and encourage a more equal relationship with students, recognising that he or she will not know the answers to all the problems and questions that arise.

The task of helping teachers to face these challenges is awesome, and models for doing so must be developed and tested. It is equally important, I believe, to prepare material which minimises such demands while retaining the key objectives; this aspect is less widely accepted. We have begun to take the latter

approach in the context of real problem solving in our Anglo-Australian collaboration. We are aware, however, that factual recall and the imitative exercise are alone known to be teachable by all teachers.

Realistic Targets

In all the above the unifying element is a plea for realism, which needs *controlled experiment* in order to establish that the targets are realistic. Since we cannot assume, or even expect to improve teaching standards much, we must accept lower targets in the sense of mathematical techniques, if we seek higher targets in terms of understanding (involving explanation, extension and even adaptation as well as better retention) and problem solving power. Let us press on to develop an experimental curriculum based on problem solving mathematics and its applications, accepting that the problems solved by most children will be much simpler than those they have been able to tackle by inaccurate imitation of their teacher's model solutions.

The Problem Curriculum

Now to the hopes and ideas:

1. Mathematical techniques – problem solving involves a greater concentration on strategic skills which requires of the student a realist view of the skills he has and his level of competence in each. He will rely heavily on his *mastery zone* – the things he knows he can get right – amid his much wider *zone of acquaintance* in mathematics. In teaching techniques we should try identifying such zones consciously – the mastery zone will need to be quite small if we are not to fall back into the present pattern of a drill dominated curriculum. The ability to *recognise* and *remedy* errors, which research suggests is more important than learning correct methods, also needs attention. The range of techniques taught could change to include more specific teaching of translation (words = graphs, etc.) and symbolisation skills.
2. Mathematical problem solving will be a central activity, but since it is the centre of others' attention and since all the problems it raises, except for that of conveying the convention of an artificial 'game', also arise with real problems and other applications, I shall not discuss it further here.
3. Real problem solving involves a wide range of skills in formulation, solution, interpretation and validation of the mathematical model of the situation of interest. Little is known of the constituent skills involved or how to teach them, though there is some research[2] and there are some heuristic methods, not unlike those of Polya for mathematical problem solving, which have been shown to be successful with volunteer teachers. The concrete practical situation involved and the acceptance of practical criteria of success in modelling partly compensate for the extra demands. The field is reviewed by the present author in *The Real World and Mathematics*.
4. The USMES Project (Unified Sciences and Mathematics in Elementary Schools)[4] was much the most extensive project so far, involving over 50 000 children in primary schools and a smaller number of secondary schools.

Its challenges were envisaged as class projects lasting several weeks. It supported the teacher by being strategically prescriptive as to the methods and problems to be used, through teachers' guides on each 'challenge' which included 'logs' of other teachers' experiences. USMES also had an extensive program of in-service training courses. It implied a substantial commitment on the teachers' part. The recent Open University course *Mathematics Across the Curriculum*[5] develops the same approach, adding a further set of examples on a smaller scale than the USMES ones. The 'starter pack' from the Shell Centre *Beginning to Tackle Real Problems*[6] aims to give the interested, but uncommitted teacher, an opportunity for a short trial of this approach. Other material[7] is being developed for other teaching situations including work with lower ability pupils in science and technology.

5. By introducing exercises of specific modelling skills (generating lists of factors, selecting model relationships, generating relationships, posing specific questions, estimating, validating, etc.) with model answers and marking schemes, the material in reference aims to provide a feeling for teachers of more conventional teaching activity with more easily measurable success, to balance the less familiar challenge of complete open problems which are necessarily involved. Exercises on various translation skills with words, pictures, graphs, tables and algebraic expressions successively representing situations are also being developed at the Shell Centre[8] on a similar realistic basis. These approaches have been tested with low ability children as well as the more able. Through the processes of devising models as well as through model selection, the student comes to the important fundamental recognition that mathematics is an artifact, involving many choices of convenience and taste on criteria based both on usefulness and on elegance.
6. The calculator and the microcomputer are proving useful aids to the pupil and the teacher, respectively, in the main. Micro-computer based teaching units are much more effective than printed material in conveying to the teacher essentials of the *style* changes that real problem solving implies – this in-service education role, using the strength of the 'personality' of the microcomputer, may prove to be its greatest short-term contribution.
7. The development of classroom methods and materials is the central task of this exciting idea for curriculum development. Banks of problems, and of exercises in modelling and other skills, will be needed and these will be slow to develop at the level of quality we shall ultimately require. Experience with computer materials has led us[9] to an approach to classroom development, emphasising the *need to observe a wide range of teachers* with *different styles* using the material with the minimum of special support from the development team. Similar methods are likely to be appropriate to this new approach; teacher accessibility will determine its success.

We shall need luck and brains, as well as stamina.

Notes

1. *Concepts in Secondary Mathematics and Science*. Centre for Science and Mathematics Education, Chelsea College, 1980, gives a detailed study of levels of competence.
2. Treilibs, V., Burkhardt, H. & Low, Brian. *Formulation Processes in Mathematical Modelling*. Shell Centre for Mathematical Education, University of Nottingham, 1980.

3. Burkhardt, H. *The Real World and Mathematics*. Blackie and Son, Glasgow, 1981.
4. USMES publications available from Moore Publishing Company, Durham, South Carolina, USA.
5. *Mathematics Across the Curriculum* (course PME 233). The Open University, 1980.
6. *Beginning to Tackle Real Problems*. (Draft version). Shell Centre for Mathematical Education, University of Nottingham, 1980.
7. See list of publications of the Shell Centre for Mathematical Education, University of Nottingham (available by post).
8. Swan, Malcolm. *The Language of Graphs*. (Draft version). Shell Centre for Mathematical Education, University of Nottingham, 1980.
9. Burkhardt, H., Eggleston, J.F. & Fraser, Rosemary. *The Classroom Development of Teaching Materials*. Shell Centre for Mathematical Education (in association with the Council for Educational Technology), University of Nottingham, 1981.

Problem Solving: Changes Deserving Attention

Alan Osborne

Preamble

Advocates of almost every curricular reform in mathematics education have touted their reform as a means of increasing learners' performance in problem solving. On the one hand, this type of rationale is testimony to the integral part problem solving plays in mathematics and its applications. The belief in problem solving as a goal appears well justified if one relies on the statements of rationale for the assorted and varied curricular reform movements of the past eighty years. On the other hand, the implementations of the curriculum ventures of the past have apparently not added up to significant progress in helping learners be more proficient in problem solving. If the curricular ventures had lived up to the promise of their rationales, would we be finding the extensive attention to problem solving that characterises curricular redesign that is currently being considered in a large number of countries? This suggests that some critical components of the teaching/learning process for problem solving have been ignored or overlooked in the thinking about instructional and curricular decision making for mathematics.

The purpose of this paper is to identify some components of the teaching or learning of problem solving that should receive more attention if we are to make curricular reforms in mathematics that will have a continuing or lasting impact. The candidates for increased attention are labelled:

1. Habituated behaviours
2. Memory
3. Heuristics
4. Getting started
5. Mathematisation

Each of these candidates for increased attention is discussed and then a concluding section reflects on characteristics of problem solving in terms of today's goal of increasing the emphasis on problem solving in the curriculum.

Habituated behaviours

Some of the mathematics curriculum is directly concerned with problem solving. Learners encounter situations involving mathematics that have a goal state (the solution) that they desire (we hope) but for which they do not have knowledge of a routine process or algorithm for progressing to that state. However, the majority of students' experiences in mathematics are not of this problem

solving nature. Rather, the experiences are with application of previously learned routines and algorithms. The latter type of experiences we will call exercises in contrast to the problem solving described earlier.

The realist recognises that the majority of encounters of learners with mathematics will remain of the exercise variety in the coming years. I would argue that this is not altogether inappropriate in that there are a few mathematical routines and algorithms that are sufficiently fundamental in both scientific and lay uses that they deserve attention extending beyond experiences in problem solving settings. For example, the distance-rate-time relationship is so widely useful that it warrants considerable instructional attention in situations above and beyond problem solving encounters.

The continuing emphasis on exercises in the mathematics classrooms instead of problem solving suggests that we need to examine the contribution of exercises to the problem solving process more carefully. The exercise setting can provide a context for establishing habits and expectations about doing mathematics that are critical to problem solving and can reveal some of the nature of mathematics. Below, two critical factors concerning learning to solve problems are examined in terms of being established in the exercise setting.

The first factor concerns looking beyond the statement of the problem or of the goal for the means of problem solution. Most problems require that the solver find an idea or a process that is not explicitly stated or contained as a part of the problem. Do the exercises that our students encounter lead them to look beyond the problem statement for ideas to use in constructing the solution? A judicious selection of exercises with an appropriate discussion can lead to a habituated behaviour or expectation that a means to solution can be found external to the problem. The majority of exercises of most elementary programs focus on computational algorithms. Computing the difference of 432 and 294 requires the student only to examine the statement of the exercise and use previously developed concepts of place value and regrouping. Exercises of this type do not lead a child to the expectation that an idea external to the problem statement will be useful in finding the answer.

For the young child in the early primary years, exercises can be designed to create the expectation that using mathematics from other than the goal statement can lead to problem solution. For example, I suggest that the child should encounter a variety of exercises of the following type:

1. If $6 + 3 = 9$, then $9 - 3 =$ ___
2. If you know that $653 + 26 = 679$, can you use that to find the sum of 656 and 29?

The use of the conditional language provides a structure of supportive linguistics for the teacher and the students while using the one bit of information to deal with the exercise. The deductive syntax of the exercise is undoubtedly going to contribute to children's refinement and elaboration of basic principles of logic but, in my opinion, this is not nearly as significant as the development of the expectation of finding leverage for solution that is external to the goal statement of the exercise. Finding and using many such exercises in the instructional sequence can help children habituate the behaviour of looking beyond the goal statement in problem solving for the means of solution.

For older learners the provision of the explicit statement of the external cue to working the exercises should be withdrawn and the instructional emphasis should shift to encouraging children to seek ideas beyond the goal statement

of the exercise. The teacher has the twofold responsibility of providing ample exercises of this type and of encouraging the verbal interaction that leads to the valuing of the behaviour of using mathematics external to the goal statement of the exercises. As an aside, it should be noted that this emphasis in doing exercises has an impact on performance in computation. It is consistent with the instructional approach described by Thornton (1978) which emphasises thinking strategies in teaching the basic facts, an approach that research has indicated contributes significantly to computational proficiency.

The second of the characteristics of instruction on exercises that needs thoughtful elaboration and careful consideration in curricular design concerns strategies. Polya's *How To Solve It* (1945) established the current identification of strategies or heuristics as what should be taught and learned in problem solving. Mathematics educators have accepted heuristics or strategies as the goal of instruction on problem solving. Yet the majority of exercises that learners encounter in school mathematics has little to do with strategies. If the majority of encounters of students with mathematics consists of exercises, it would seem to be common sense to design and include in the exercise sets many activities that concern strategies. The encounters should be emphasising a notion of what a strategy is, how one judges the effectiveness and power of a strategy, how to recognise a strategy, and the like.

Given all of the strategies that can be used in mathematics, is it wise to reserve for problem solving activity the sole attention to strategies in mathematics? Children learning about strategies in thinking currently must not only deal with problem solving but at the same time learn about strategies. If strategies or heuristics are the important objectives associated with learning about problem solving, then it is important to provide learners with judgment about strategies and heuristics prior to and independent of the problem solving experience. Further, it is important to establish the habituated behaviour of thinking in terms of strategies when doing mathematics.

The curricular task of increasing the emphasis on strategies in designing exercises consists in large part of locating topics in mathematics that are natural settings for learning about strategies that at the same time are important basic skill areas for the school mathematics program. I would suggest that estimation of measurements and approximate computation are topics that would serve the purpose well. These topics are an important domain of basic skills and provide a setting for developing knowledge of and judgment about strategies.

Suppose that a learner is to estimate the answer to the computation exercise 33×78. Strategies that might be encountered are:

A. $30 \times 70 = 2100$
B. $30 \times 80 = 2400$
C. $\frac{1}{3} \times 78 \times 100 = 2600$
D. $35 \times 2 = 70$; therefore, $35 \times 4 = 140$; therefore, $35 \times 8 = 280$; therefore, $35 \times 80 = 2800$.
E. $35 \times 80 = 70 \times 40 = 2800$ but both 33 and 78 were rounded up and the answer is too large by about two 80's; therefore 2640.

These strategies are relative in their effectiveness and power to deal with the estimation problem expeditiously. They provide a comparative context for students to learn about strategies that yields a readiness to deal with strategies and heuristics in the problem solving setting. Significantly, focus on approximate computation strategies can establish the expectation that strategies in thinking is a part of mathematics.

Hildreth (1980) has interviewed children and young adults concerning estimation of length and area measurement. Both children and adults exhibited strategy use. If strategies were used, the accuracy and rate of estimation were significantly better than if strategies were not used. Although no qualitative differences in the strategies used by young adults and thirteen-year-olds were observed, he did find that some children had difficulty in verbalising what they had done to produce an estimate. The implication for teaching is that for the teacher to help children acquire psychological closure on a strategy, the requisite linguistic register or 'official' language about strategies needs to be possessed by the learner. This register can only come through experience. Exercises concerning estimation provide an appropriate setting for acquiring this register that effectively extends and elaborates the instructional emphasis on strategies and heuristics in problem solving. If learning strategies and heuristics is the *sine qua non* of the curriculum in problem solving, too much is at stake to reserve only for problem solving activities an emphasis on heuristics and strategies.

Memory

One of the more intriguing academic and intellectual puzzles in the field of mathematics education is the lack of attention to the factor of memory learning and doing mathematics. Researchers have given very little attention to this fundamental process of the mind in studies of mathematics learning. Except for the perfunctory retention tests associated with mathematics methods experiments, little has been done to examine how memory works or how it contributes to success in activities such as problem solving. Particularly ironic in an era of concern for performance and basic skills, how much do we know about the role of memory in problem solving and the effect of problem solving on memory?

Krutetskii (1976) suggests that children with ability in mathematics have more elaborate and different memory mechanisms than those with lesser ability and that this ability is demonstrated in problem solving encounters. The high ability child has a greater range of cues that are used to trigger the memory processes than the low ability child. A number of researchers in learning theory have demonstrated that if the learner possesses a cognitive structure, a schema of inter-related ideas, then retrieval of ideas is facilitated. Learning of mathematics in discrete, separated chunks does not facilitate the processes of memory or the retrieval of cues that serve to initiate the reconstruction process of memory if the learner fails to recall what needs to be remembered.

If the act of problem solving substantially contributes to memory, then this constitutes one of the more powerful arguments in support of increasing the emphasis on problem solving in the mathematics curriculum. Evidence of the few experiments in this area suggests that problem solving does contribute significantly to improving learner's memory of mathematics. Duren (1980) examined children's ability to recall and to reconstruct four rules in mathematics as a function of instructional or methodological variables. Three different treatments were used: problem solving via an inductive teaching style, problem solving via a deductive teaching style, and rule-example practice. Data from this carefully designed experiment indicate that for those students who had initially learned the rules, after a six to eight week period of no practice, those who had been taught via problem solving exhibited significantly increased ability to recall the rule and significantly improved ability to reconstruct the rule if they did not recall it. The significance was not only statistical but also educational. Duren's

work also provides some weak but promising indication that instruction on problem solving can be supplemented by instruction on how to remember, that further aids the learner in remembering rules. Thus, we have some evidence that indicates that more is at stake in teaching problem solving than increasing children's ability to solve problems; problem solving also helps children remember mathematics.

Frequently the barrier between a mathematical situation and the solution or goal statement of a problem is memory. The problem solver does not remember the process or find a cue that triggers the reconstruction of the process that will yield solution. Indeed, recall or reconstruction often reduces a problem to an exercise since it does remove the barrier between the situation and the solution. The most powerful statement that can be made concerning memory and problem solving is that the more one remembers, the more likely that one is a good problem solver. I am often astounded by the apparent trivial level of the mathematics that is remembered that triggers the solution process in problem solving. In an era of concern for and commitment to teaching strategies and heuristics, it is appropriate to observe that learners need to acquire a large store of 'facts' since they serve the function of cueing the memory process. In my judgment, a curriculum that emphasises problem solving will not concentrate exclusively on teaching heuristics and strategies for problem solving but will also focus on helping learners acquire the store of 'facts' that constitute the base knowledge and intuition to be used in the problem solving process.

The nature of memory processes and how they work in problem solving is a domain of research in mathematics education that has been overlooked and ignored by professionals in the field. Having profound and significant potential for improving instructional and curricular practices in the classroom, research in this domain should receive high priority from mathematics educators. In the meantime, teachers of mathematics should put in practice what is known about helping children remember. Knowledge concerning the effects of distributed practice, scheduling reinforcement activities and the like does provide some effective guidance for how to help students remember. Most importantly, instruction cannot be solely concerned with processes of problem solving but must help students acquire a cognitive structure that serves to integrate mathematical ideas and facts.

Heuristics

The literature of mathematics education relevant to problem solving, whether concerned with instruction, curriculum or research, focusses attention on the heuristic strategies of problem solving. These strategies are identified as suitable objects of teaching and learning. The focus on strategies appears to be well-justified in the sense that strategies are processes that can be employed whatever the problem situation. The use of strategies in solving a problem amounts to attempting one strategy; if it works, you arrive at the solution but if it does not, you select another strategy to test. Some strategies, such as drawing a picture to represent the problem situation amount to little more than practical good sense discovered by scores of learners and problem solvers in many generations. Other more elaborate strategies, such as described by Wickelgren (1974) in terms of state spaces and intermediate subgoals, are more complex, detailed and do not possess the apparent 'natural' character that will make them easy for learners to appreciate and use. Mathematics educators need to recognise that we have little to guide us concerning the design of learning experiences appropriate for teaching problem solving strategies. The judgment concerning the 'naturalness' and ease of learning the strategies rendered above is an a

priori judgment. Our lack of experience in teaching strategies means that many important curricular and instructional decisions must be made on a basis of rational analysis of the content to be taught and an educated guess about the reaction of students. Thus, in terms of curricular development, we are in the exciting position of venturing forth in new territory but have the profound risk of making a few mistakes.

Several curricular concerns are implicit in mathematics educators' attempts to shift the curriculum to a greater emphasis on problem solving and the learning of problem solving strategies. Since the concerns will undoubtedly result in elements of discomfort by teachers who must implement such curricula in the classroom, they deserve attention. The concerns are identified as questions that need to be addressed with the expectation that mathematics educators will learn enough to modify the curricular approach if the evidence warrants.

Question 1. What strategies are most suitable for instruction?
Several descriptions and lists of strategies for problem solving are available (see Krulik [1980], Schoenfeld [1981], and Wickelgren [1974]). Should all such strategies be taught? What are the criteria for selecting strategies to be included in the curriculum of school mathematics? Some strategies are quite global (analyse, plan ,explore, test, verify . . .); others are more narrowly based and do not apply to all situations. Are the global strategies 'best' for all students? In short, in terms of the traditional curricular criteria, what should be the scope of the strategies contained in the curricula of school mathematics?

Question 2. When should particular strategies be taught?
Few school systems in planning curricula for problem solving have developed a sequence for teaching problem solving strategies. Any strategy that can be exemplified with a problem that is understandable by children is fair game as an instructional topic. The almost total reliance on the characteristics of the problem situation for guidance in emphasising heuristics at given age or grade levels means that children may encounter some heuristics over and over at several different points in their schooling but may miss others. It seems reasonable to advocate that curricula concerning problem solving heuristics should identify particular strategies to feature corresponding to different stages of mathematical maturity or grade levels. This would assure a comprehensiveness in the problem solving curriculum. But do we have suitable insight about which strategies are most appropriate at different stages of mathematical development? For example, at about what age will children benefit most from experience with the strategy of restating the problem in their own terms?

Question 3. What are effective instructional procedures for teaching strategies?
My experience has indicated that to teach a problem solving strategy I must find a suitable problem, guide students to the use of the strategy, and then help students look back to the problem solution in order that they may identify the strategy. Discussion of the process of solution seems a necessary ingredient to helping students attain a psychological closure on the strategy and an appreciation of the power of the strategy. I am quite confident that the thinking about the thinking process is necessary if strategies are to be the objects of instruction. But there is very little research evidence that I find that supports my personal convictions. Simply exhibiting the strategy does not suffice; students need to indulge in reflective thinking about a strategy. Talk and discussion

concerning the strategy focusses attention on the value of the strategy. Schoenfeld (1981), in comparing the problem solving behaviour of experts and novices, reports results supporting my conjectures and comments about underestimating the amount of instructional time necessary to establish heuristic strategies. In brief, teachers can encounter considerable frustration in trying to teach strategies since we do not have much refinement of instructional techniques suitable for heuristics.

Question 4. What are suitable problems to serve as vehicles for establishing particular strategies or heuristics?
Most individuals come to an appreciation of a strategy through encountering a particular problem that a strategy fits. We need to develop a catalogue of problems indexed in terms of the strategies that fit well in terms of pedagogical leverage. Further, it would be helpful if the catalogue were to document the performance of learners concerning the strategies.

Doyal Nelson (1976) in discussing a research project concerning problem solving in early childhood identifies seven criteria used for selecting problems to fit the model of thinking studied in the project. The first criterion is that the problem should be of significance mathematically. The current emphasis on strategies raises the spectre of problems being selected only on a criterion of their effectiveness in establishing strategies. Problems should also be considered in terms of Nelson's criterion of involving significant mathematics. This will also protect us from developing a problem solving curriculum that is too oriented to the novel, cute and out-of-the-ordinary.

Question 5. How do we evaluate the acquisition of strategies by learners? Most of us work diligently to construct test instruments for the skills, concepts, and principles of traditional mathematics. The challenge of creating evaluation of strategies for problem solving will be significant because many of the usual paper and pencil devices are not appropriate for ascertaining whether a student possesses a strategy or not. And we know that learners, particularly at the secondary school level, rapidly figure out which objectives that are being taught are being tested; they quickly deduce what types of objectives deserve investment of study time.

Getting started

The most fundamental behaviour in problem solving is initiating the solution process. Trying something even though it may be wrong must be done, otherwise no solution will be made. Perhaps the largest irritant to many teachers in trying to help students acquire problem solving strategies is the passive student that makes no venture about the problem. Curriculum developers and teachers particularly committed to the teaching of problem solving must find and develop specific instructional strategies that encourage learners to attempt solution. Relying on the attractiveness of the problem is not sufficient because learners vary dramatically in what they find attractive and interesting. The problem solving behaviour identified by some authors as a strategy of making your best guess works well with some students. Making the guess provides the minimal threshold behaviour needed to involve the student with the problem and frequently leads to solution in that the act of making the guess be the 'best' helps the solver identify critical aspects of the problem.

Another aspect of start-up strategies in problem solving is that they involve confidence as well as the motivation of the learner. Good problem solvers tend to exhibit a supreme self-confidence in their ability to come up with a solution. Often I note among my friends that this self-confidence extends beyond the realm of mathematics; they are willing to tackle any problem or challenge. If we are to be successful in teaching problem solving, must we not also find ways to generate this self-confidence in our students?

Mathematisation

Ultimately, learning must serve a useful purpose. Otherwise, the individuals and agencies that support the schools and the students will limit their commitment to education. For mathematics, this translates to 'mathematics must be useful'. Thus, a curriculum concerned with problem solving must also be concerned with the uses and applications of mathematics. Mathematics purely and simply for the sake of mathematics is not a sufficient justification for a place in the curriculum even though some products of our mathematics programs should become committed to mathematics for its own sake. I would argue that a considerable amount of the problem solving activity in classrooms at most levels of school mathematics should be devoted to mathematising the real situations that are of potential significance to adult life in both the sense of the layman's uses of mathematics and the applications of mathematics that characterise modern business and science.

Two particular curricular thrusts need to be considered in developing a problem solving emphasis in terms of applications. First, students need experience in looking at situations and finding problems within them. Too often children and youth have no school experience that helps them discover problems. All of their problem solving activities are stated completely in that the solution goal is stated for them. Students need experiences in which they are given a situation and asked to find a problem to solve (a goal statement) appropriate to the situation. Parenthetically, it should be noted that there are some strategies for identifying problems that could be taught such as modifying one or more conditions of the problem, adjoining another variable, reasoning by analogy, and the like.

Second, students should encounter the process of mathematical modelling. This process serves to distinguish modern science from the science of even a century ago and is among the more powerful intellectual tools currently characterising scientific thought. The experience with making mathematical models should begin early in the curriculum. A child who writes an equation that tells a story of a verbal problem is making a mathematical model of the problem. Often we have protected children from mathematical modelling because we have considered modelling primarily in terms of complex examples for which recognising major assumptions and specifying variables with some degree of completeness is at a premium. But initial experience with modelling needs to begin early with simple examples. Secondary school students should encounter modelling in examples that do increase the level of complexity and involve such fundamental characteristics of the modelling process as the recognition of assumptions and specification of variables as well as the predictive testing of the model.

The word mathematisation, the title of this section, is a coined word that I believe is suitable in considering applications of mathematics. The intent in using the word or its verbal form, mathematising, is to communicate that applying

mathematics is a constructive, creative act that is part of doing mathematics. The phrase applications of mathematics separates mathematics from application and does not communicate the constructive, active atmosphere that I believe should characterise children and youth learning to use mathematics.

Concluding discussion

I am optimistic about changing emphases in the mathematics curriculum to increase the attention given to problem solving. Not only is problem solving fun, if students can acquire the capability to deal with situations about which they have not been taught and for which learned rules do not apply, they will have acquired a significant personal power and command over their futures. Clearly, a dramatic shift to the teaching of problem solving heuristic strategies will represent a major reorganisation of our thinking about curriculum and instruction. Problem solving strategies have not been considered in traditional curricular terms of sequence, scope, readiness, evaluation and the like. As professionals in mathematics education, we need to recognise that we are entering some uncharted waters and that we should be taking stock of our path in order that we can identify expeditious routes for instruction. We also need to recognise that some candidates for curricular ventures may, like the quest for a northwest passage to the Orient, turn out to be non-productive on exploration.

Part of my rationale for stressing exercises in the first section and memory in the second section was to associate problem solving with factors that have in the past been highly valued in mathematics teaching and learning. The history of mathematics education indicates that exercises and memory have been considered important and valuable. I suspect that currently many lay people as well as teachers associate exercises with mathematics in the classroom and will continue to do so. Developing a curricular emphasis on problem solving that ignores the contribution of exercises to the acquisition of problem solving is probably an unrealistic route to change. Exercises are too ingrained as a traditional part of classroom practice to separate them from the teaching of problem solving. Memory, a fundamental cognitive process, needs to be incorporated into our pedagogical orientation to problem solving. More is at issue than the fact that memory cues or triggers problem solving processes. Individuals who judge the effectiveness of the school attribute high value to the school serving as a knowledge producer for students. This is assessed partly in terms of memory of the products of the school. The contribution of problem solving to generating memory is significant and important.

Problem solving plays a central role in mathematics. I consider it to be at the heart of mathematics. Yet, it is but a part of mathematics. The majority of mathematics done by individuals, be they lay people or mathematicians, is not accurately described as problem solving. It is more accurately identified with the use and application of ideas and structures. Generalisations are made and used without encountering the barrier to thought and solution that characterises a problem. It is wishful thinking to hope that we are preparing learners for a mathematical life constituted totally of problem solving. Indeed, the majority of our students will use mathematics in ways that we might aptly describe as mundane if judged on a criterion of whether problem solving is involved. We run the significant risk of over-promising the payoff of mathematics study if the rationale for a curricular shift to problem solving is based on the prospects of students maturing into a life totally filled with problem solving.

I am concerned when I examine rationales for increasing the emphasis on problem solving in the curricula for mathematics and for other disciplines. The primary point of many rationales is that acquiring an ability in problem solving is necessary to students in order that they may deal adequately with the unexpected in their future. Focussing on the rapidity of change and the apparent acceleration of the change process, the writers justify the shift to problem solving in terms of dealing with the unexpected and the unfamiliar. This narrowly conceived justification for problem solving is symptomatic of an apprehensive psychology that is concerned primarily with coping rather than the creative development of new ideas and new mathematics. This rationale may carry a subtle characteristic that will interfere with achieving the curricular goal of helping students acquire problem solving abilities. The apprehensive psychology is inconsistent with the psychology of self confidence and building a faith in problem solving capability that characterises many good problem solvers. If we accept this narrow rationale for problem solving and develop a curriculum totally oriented and justified in terms of the unexpected and unfamiliar, then we risk modelling for our students attitudes that are contrary to the confidence that is necessary for problem solving. Further, if students encounter only problem situations and contexts in their mathematical experience in schools, do we run the risk of interfering with the development of confidence in our students? The shift to a curriculum more directed at problem solving needs to be justified on a broadly conceived basis that respects the role that problem solving plays in mathematics and in learning mathematics.

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Problem Solving in Mathematics in Australian Schools: Thoughts, Views and Comments

Bruce Henry

The word 'problem' is used in different contexts in different places. Some North American mathematics educators would regard any mathematics question written in words as a 'problem'. This definition is far too broad. I usually expect a problem to be a question which cannot be solved using a standard approach known to the solver. This means that a question may be a problem to one solver and not a problem to another. For example, a problem which can be solved by the setting up and solution of a pair of simultaneous equations is a problem to a grade five child who has no knowledge of equations, let alone simultaneous equations.

I include in my definition of a problem a question in which it is necessary to apply two or more standard, but previously unrelated procedures, to arrive at a solution. For example, an average and all the numbers but one are given and the question requires that the last number be found. I would also include pattern-seeking types of problems which may be solved by the development of a pattern or a systematic counting process. For example, for a student with no knowledge of permutations and combinations, a problem which can be solved by determining ${}^{6}C_{4}$ should be classed as a problem. Clearly, a problem for which there is no routine method of solution will be classed as a problem in the sense we are using the word here – the solution of 'Instant Insanity' or Rubik's Cube is clearly a problem.

To my knowledge, there are three, not necessarily distinct, methods by which students are likely to acquire an ability to solve problems. I also believe that some people do not need to be taught, but have an innate ability, which nonetheless may be improved by instruction and practice. The first of these is the well-known method of teaching and practising 'problem solving skills' of which the best known are those popularised by Polya (1945) in his book, *How to solve it*. The second is a method more frequently applied to problems outside mathematics, most easily described as 'brainstorming', where the solver is encouraged to think of as many widely different ways of approaching the problem as possible. This will involve the solver in asking many questions about the problem. Many of these questions may be unrelated to each other, but may help to clarify the problem or lead to a method of solution. The third method is to teach by example: the teacher poses a problem and has a successful solver tell the rest of the class what thought processes led him to the solution.

On the first method, John Malone's research in WA gave a surprising null result. This has led me to think that students have largely managed to acquire the skills being taught for themselves by the time they reach year 9. Thus it would be interesting and probably informative to repeat Malone's work with much younger children, for example year 5.

It seems to me that there can be no harm done in teaching problem solving skills in mathematics courses, provided the teaching is good teaching and the

problems selected of the correct level of difficulty. Surely everyone would learn something from such a course; whether any transfer occurs to the solution of problems in real life is open to question. There is clearly quite a large area of grey which needs to be carefully researched into black and white. I think the statistical and experimental techniques are available, but the job is a very large one.

Even less is known about brainstorming. Again, it seems to be harmless enough to teach it as an approach to problem solving. I do not think it should be the *only* method taught. I know of no research on this method, yet it is used widely in the business world, in management courses and, generally, as a problem solving approach in a non-mathematical situation. Can it really be applied to mathematically flavoured problems with an expectancy of good results? Again, here is a large, woolly area which is going to be a difficult one to clarify by research. Research is going to involve an experiment which must have a fairly long duration if the approach is to be given an honest test.

Most teachers use the third method, many of them unconsciously. Many students learn what problem solving skills they have by this third method, again, many of them unconsciously. There is next to no research that is readily accessible to teachers. I have already suggested that I think children have learnt the problem solving skills by Year 9, and this must be the way they have acquired them, given that they needed to acquire them and did not already have them.

Certainly, more information is needed. First and foremost, we should collect information about the natural development of problem solving skills by children of various ages and abilities. When this is known, and it probably relates to Piagetian stages somehow, we need to find the ways by which children acquire these skills and to what extent this acquisition can be accelerated or enhanced by teaching – and by what kinds of teaching. I believe the road to be a very long one and one on which we have hardly set foot. We are back in the days before Piaget and much basic research is needed. Meanwhile, teachers need a guide to help them teach children the necessary problem solving skills. Children need to be guided through a series of suitable problems so that they can practise, establish and improve their problem solving skills. Hence, at the same time as research is being undertaken by those qualified to do it, curricular development needs to take place to provide teachers with the wherewithal to treat the problem now. Suitable learning materials, teachers' materials and strategies need to be developed and made available. Even if these materials are initially only in trial form and are modified as a result of use, they will serve a purpose in filling a void in mathematics curricula.

Thus I see a two-pronged attack on the problem as the *start* of a solution to it: research to find out more about the problem solving processes and the approaches to teaching problem solving, as well as the development of materials which the teacher can use in the interim until more is established by the research. This material development will also be a form of research and may well provide some of the answers. This attack will need to be co-ordinated, so that the limited resources are used in the best way. This co-ordination seems to be an appropriate role for the Australian Mathematics Education Program.

Problem Solving with Mathematics in the Real World

Christopher Ormell

It is widely agreed that much of what passes for 'mathematics education' in schools consists in getting children to learn algorithms. Many mathematical educationalists from Whitehead onwards have recognised that this is far from satisfactory.[1] Yet very little has been done to remedy the situation. The cataclysmic changes associated with activity methods, child-centredness, discovery learning, individualised schemes, even 'New Maths', have come and gone and, at the end of the day, most teachers in most schools still seem to perceive their central task as 'getting children to learn algorithms'.

The reasons for this sorry state of affairs need to be fully understood if we are to make any progress towards a healthier system. One underlying reason seems to be that until recently *industry* has been, or has been supposed to be, behind the existing system. Today industry, too, is beginning to see that the diet of algorithm-memorisation which has been served for so long in schools is singularly useless in training children to think.[2] Rather than promoting the child's capacity to think it probably inhibits it. The child is encouraged to 'memorise and then apply' algorithms, but no thought is required in the memorisation, and little is usually required in the application. This is a comment on the types of 'applications' used. Most of the so-called 'applications' of the algorithms learnt are simply cued performances of them. For convenience in the sequel, we shall refer to this kind of production of an algorithm-on-cue as *handle-turning* work.

This, then, is the background against which one may see the recent wave of interest in promoting problem solving in school mathematics. The case for using some sort of 'problem based' approach in school mathematics now appears to be quite overwhelming: both because the alternative is visibly obsolete, and because getting children to solve problems is clearly a worthwhile, positive aim. To 'have problems' is uncomfortable. To be 'able to solve them' is just what everybody wants! Indeed, the case for a problem based form of school mathematics is now so irresistible that one may wonder why it has not long since come about. What is holding us all back? Why is not problem solving already universal?

The short answer to this question is probably that 'problem solving', as such, is not a distinct idea. It is, in effect, defined negatively, i.e. as work involving less than 100% handle-turning. It is said that there are a thousand ways of being unhappy, but only one way of being happy. Conversely, there are a thousand kinds of 'problems' and, hence, ways of solving them. This leaves open the question: in what does an *educationally viable* problem consist? Or, to put it another way: what kinds of problem is it reasonable to expect children at a given stage of development to solve?[3] Unless we can answer these questions in a fairly detailed, constructed way, the problem based curriculum is likely to remain a distant ideal, rather than a working reality. It is comparatively easy to set children problems. The difficult thing is to get children to solve them! Sometimes –

at odd moments – in the classroom, one feels a great sense of satisfaction arising from the fact that the children have actually managed, unaided, to solve a particular problem.

This is a breakthrough to real thinking! But it should be pointed out that the 'problem based curriculum' will require this miracle repeated, not once or twice, or even a few times, but right through ten or twelve years of progressively more complex work. This is a measure of the difficulty of the task. The difficulty we face is like that of the early aviators, who were always at the mercy of a stall. It was not hard to get off the ground and to fly in short hops. What was difficult was to fly in a secure and sustained way, purposively from A to B.

If we are to tackle the central problem of identifying and defining what is an educationally viable problem, we shall need a vocabulary with which to describe such problems. This, in turn, will need to rest on an account of problem solving as an experience in general, and as an educational experience in particular.

We have to steer a tricky course between the Scylla of easy handle-turning and the Charybdis of endemic difficulty – of demoralised children, deflated teachers, disappointed examiners. The basic truth is that it is extremely easy to set problems which are too difficult, or which *seem* to be too difficult.[4] And problems which are too difficult can shake children, even talented, thoughtful children, to the roots. In this way it is possible in a single afternoon to lose much of the confidence and momentum which one has built up patiently over weeks of rolling, constructive work. What usually happens is that the teacher steps in and 'does the problem for the class'. It is best not to dwell on the message such an episode conveys: its implicit statement that the children are not up to solving problems for themselves, that they need to be shown the trick, and that this is their natural, passive role.

This paper is mainly concerned with the question of the definition of viable educational problems involving the application of mathematical models to real situations. Opinion is divided at the present time over whether such real life assignments form the major or the minor branch of the problem solving genre. Cogent arguments have been advanced in recent years for regarding real life problem solving as the major branch, but this is, it must be admitted, quite contrary to previous assumptions on the subject.[5] It takes time to absorb this new approach. Some of the analyses included in this paper bear on the question, and, in the author's view, add still further reasoning to the already weighty case for the major role. But the question of the relative significance which should attach to real life, as opposed to formal, problem solving is not, as such, part of the remit of the present paper.

What is 'a problem'?

The first task is to achieve a clear view of what is meant by 'a problem', 'formulating a problem', 'solving a problem', etc. This is fortunately a fairly straightforward task: in fact, definitions of what we mean by 'a problem' have been given by various authors. Ann Taylor (1970) suggests 'A problem may be said to arise whenever a desired goal is not immediately accessible.'[6] More generally, people may be said to 'have problems' when they cannot *see* the way to reach some goal state. Two examples follow. A government has a 'political problem' when it cannot see how to persuade its followers that a certain policy is necessary. A person has a 'financial problem' when he or she cannot see how to meet his or her financial commitments.

Sometimes people speak as if the *obstacle* preventing easy access to the goal state were, itself, the 'problem'. But as soon as we *see* a way round the obstacle the problem disappears. Thus, it is hardly plausible to maintain that the problem consisted in the presence of the obstacle. Sometimes people speak as if it is not the failure to see the route to the goal state, but the failure to reach the goal state, which constitutes the problem. All that is needed to accommodate this objection is an adjustment in our concept of the goal state. The government may indeed see a way to persuade its followers that the policy is necessary – a way which takes five years! But the goal state was never a timeless one. The government's true goal state was always to persuade its supporters in a given interval of time, say before a particular senate vote. Given infinite time, most problems disappear.

Solving a problem consists broadly in searching for a route to the desired goal state.

In the case of mathematical problems the 'route' we are looking for usually consists of a sequence of symbolic and/or linguistic manipulations. Real life problems involving mathematical modelling often involve both informal linguistic and formal symbolic manipulation. It should always be borne in mind that when we speak of 'looking for' and 'seeing' such sequences we are using a convenient metaphor for the internal experience of thought, which would be, in the absence of the metaphor or its equivalent, virtually incommunicable. Evidently, too, the acuity of 'the mind's eye' – to use another version of the visual metaphor – varies greatly from person to person. What a person can 'see' in this special, analogical sense depends greatly on the quality and quantity of their prior knowledge. One might say that any general rule, Law or principle which has been thoroughly understood by the student, will tend to shed light on the kind of problems to which it applies. The light shed by half-forgotten or neglected generalisation, however, is generally much dimmer than that shed by thoroughly familiar and practised ones. Since formal mathematics consists very largely in operating with generalisations of many levels of complexity, this tendency of generalisations which are not in regular use, to decline in illuminative power in the student's mind, is crucial in determining the difficulty of a problem. In other words, different students who have followed the same course of work may differ widely in their mental 'vision'. For no two students will the problem be quite the same: the illuminative effect of combinations of laws and rules in various states of activation will vary widely: an astronomic number of different patterns can occur.

The effect of the possession of different items of knowledge has a still more marked effect on this 'mental vision'. For example, 'Express dy/dx in terms of x and y' will be a *problem* for the student who has never heard of implicit differentiation, and a one-minute handle-turning exercise for the friend who has.[7] The extraordinary variation in this 'mental vision' from person to person, and the enormous effect of specific knowledge, may seem to preclude any possibility of defining the 'educational viability' of a problem. If this agrument applies in its full force – as it does – to problems involving only the discovery of sequences of formal manipulations, how much more will it apply to problems involving both formal and informal manipulation? Again, we are brought face to face with the extreme difficulty of the challenge implicit in the problem based curriculum.

The other major factor involved in 'solving problems' is the strength of the solver's underlying desire, motivation or *drive* to reach the goal state. The central issue here is the effect that lack of success in solving a problem often has on the underlying motivation. One has to be very determined to continue to want to solve a problem which seems continually to defy solution. Motivational instability is a problem throughout education: it is the crucial issue in setting

large numbers of students difficult questions. For these students' motivational stability is thereby put at risk. Modern civilisation presents endless examples of what we might call the 'ducking out' response to a problem, namely abandoning the motivation. The response has been institutionalised in protest movements, strikes, concepts like 'lateral thinking', mottoes like 'If you can't beat them, join them!'. One might say that the trait of dogged persistence which is needed by anyone of less than exceptional talent if they are to become successful problem solvers is not a fashionable trait today. In fact, it is worse than this, since dogged persistence in problem solving is always liable to be misread by the lazy-minded as mental aberration!

The social dimension of motivation also presents difficulties. John Holt has observed that most children do not enjoy taking part in the work they do at school and from an early age they tend to adopt a defensive stance to this – that is, doing as little as possible, doing only enough to avoid incurring the teacher's displeasure.[8] Even in so-called 'progressive', private schools the same patterns can be seen. But if this is the response – as the evidence suggests it is – to ordinary handle-turning work, the prospects for adopting a problem solving curriculum are not too good! It has always been possible to do problem solving work with small hand-picked groups of able students. Trying to do problem solving work with large groups of average children is an altogether different matter.

In most schools, and for the majority of children, the experience of problem solving seems to be predominantly an unpleasant one. It is probably as a result of this experience that most children make little progress in developing their problem solving skills.

The major question we need to address is whether this must be, inevitably, so. Obviously it would be much more satisfactory if we could devise conditions under which a majority of children was able to make real progress in solving problems. But we are unlikely to get anywhere near devising such conditions, until we *first* thoroughly understand the causes of the present shortfall in problem solving education. There are several factors militating against problem solving for the majority. One is the simple fact that where a class containing a few exceptionally able children is set viable problems, the few children's success in solving them tends to affect the rest of the class adversely. First, the few tend to destroy the challenge implicit in the very problems they have successfully solved (for the other children). 'The answer' gets around the class. This effectively causes the problem to disappear for the rest of the class, because it dissipates most of the fog which initially surrounds the route leading to the goal state. Secondly, even when the answer does not become generally known to the rest of the class, much of the intensity of the corporate motivation tends to evaporate as soon as it becomes known that the problem has been solved. This seems to be a natural social reaction. As long as a problem is unresolved in a social group, it presents a substantial and glittering challenge. As soon as it is known to have been solved, the efforts of average members of the group tend to switch to trying to gain access to, and hence to share, the solution, rather than carrying on and 'working it out for themselves'. In other words, at the moment the problem is known to have been solved, the group's psychology changes from an individualised to a normal social pattern. Thirdly, after repeated experiences of this sort, the slower children in a group tend to become demotivated. So they fail to realise that it is possible for them to achieve personal intellectual satisfaction by grappling with a baffling problem and subsequently achieving a resolution of it. Instead, they repeatedly find themselves still stuck half-way through the baffling stage when the problem falls to pieces in their hands. Their problem solving inadequacy, rather than their problem solving confidence, is continually being reinforced.

One general point of some significance is the relatively short time over which the experience of problem solving occurs. For very many children success in perceiving a route to the goal state must come in one or two minutes, or it will not come at all: after two minutes of blocked vision many children's confidence begins to ebb. If they have not 'cracked the code' in two minutes, they are unlikely to crack it after that. Once confidence drains away, motivation dwindles too, and the child is simply wasting time. This is a problem bearing particularly on real life problem solving, since in this area it is not uncommon for the stage of digesting-what-the-problem-is-about to take longer than two minutes. This stage is, after all, only a preliminary to actually solving the problem.

This, then, is a brief survey of some educational, psychological and social difficulties associated with problem solving. So far we have concentrated entirely on *problems* attendant on introducing problem solving. We turn now to look at some approaches which might lead to a resolution of these troublesome problems.

What is a Viable Problem?

The issue of trying to define what we might mean by an 'educationally viable problem' is one of a fairly familiar type. In industry it is called 'establishing quality control'. It seems that this must be our first priority, if problem solving work is to make any headway in the schools. *Unless* we can establish some sort of quality control, we are likely to find it virtually impossible to introduce problem solving on a significant scale. In effect, every problem used in the average school needs an implicit guarantee:

This will not shake your, or your class's, confidence *provided* that you meet the appropriate conditions relating to the class's readiness for it.

Without such a guarantee, or something roughly equivalent, schools are likely to shy away from introducing problem solving. The risk of a serious loss of confidence attendant on using unreliable problems is just too great. Sometimes one hears the so-called argument that 'life is a jungle' and that 'it is all part of growing up for a child to have his or her confidence shaken'. But if this 'argument' is used to justify doing nothing to reduce, or neutralise, jungle conditions, it embodies a fatuous form of logic. What is important in the last analysis is that children should sharpen their intellectual teeth. They will hardly do this on milk-and-water diets. We must, therefore, present them with challenges . . . viable challenges that can be met and overcome. Or, to revert to the analogy, we need to provide them with sufficiently hard food to sharpen their teeth. It should be quality controlled because there will be howls of pain if we provide fare containing teeth-breaking lumps: like gravel in a cake.

Three straightforward conditions must apply to problems if they are to count as being 'educationally viable':

1. They should be problems which most children will feel that they would *like* to solve. We need to begin from the basis of a stable and unmistakably solid motivation. Without this prerequisite, a problem is liable to fall to pieces in the child's hand, if any kind of real difficulty is experienced in searching for the solution.
2. They should be problems which the average child *can* solve *if* he or she makes the effort. They should not embody extreme formal subtlety or

finesse, but consist, rather, in the presence of obscurities which will gradually clear as the child tunes in and works at them.

3. They should not form a special limited set, but be part of an endless stream of problems of a certain type. This is necessary in order that the questions should remain as genuine 'problems', and not simply become 'items', the answers to which can be learnt by heart.

It is clear that these conditions are not easy to meet. Indeed, it is a commonplace in education that the average child's motivation is often not sufficiently strong to justify putting 'problems' on the menu in any real sense. There is also the endemic problem that the desire to produce fresh problems (condition 3) has tended to lead to the use of ever more subtle, convoluted, unobvious formal problems, thus working *against* condition 2.

The question therefore arises: can we meet the three initial conditions at all? The answer appears to be 'Yes'.

Sometimes it is said that the greatest motivation students can feel for mathematics is an *intrinsic* attraction for its ideas, patterns and methods. Such statements contain very little information. Of course, every activity in which human beings engage possesses some intrinsic attraction as an end-in-itself. In emphasising the intrinsic attraction of mathematical activity one is not demarcating it in any way from classical languages, modern French literature, athletics or marquetry. We need a much more cogent account of the appeal of mathematics to the average youngster, if we are to make any progress in forging unmistakably strong mathematical challenges for the average child.

The author has elsewhere formulated a view of the motivational appeal of mathematics for the average child (Ormell, 1978, 1980a, 1980b, 1981).[9] It may be summarised briefly by saying that mathematics should habitually be seen as a 'possibility previewing activity' (Schools Council, 1975).[10] Peirce said that mathematics is the 'science of hypothesis'.[11] The trouble with these formulations, however, is that they do not really bring home the full impact of the type of motivation being recommended. Many people are insufficiently aware of their own cognitive and active condition to recognise the importance of 'previewing possibilities' or of studying hypotheses scientifically. This sounds rather highbrow for many people. Both terms 'possibility' and 'hypothetical' are commonly used in a *dismissive* sense to exclude courses of action and life-chances from a practical agenda. We say things like 'Ah yes, that is a *possibility,* but . . .', and 'Oh, but that is a purely *hypothetical* question'. One might assume that in writing for fellow educationalists one would be understood in writing in an enlightened way about these matters, i.e. taking possibilities and hypotheses more seriously than is the norm in ordinary society. This assumption is not, however, correct. Many educationalists cleave to the mores of ordinary society, and go along with the implicit anti-intellectualism of the dismissive attitude to possibilities and hypotheses. It appears to be necessary, therefore, to formulate the view of motivation advocated here – as a basis for strong problem solving – in a rather different light.

First, it needs to be said that the motivational appeal of mathematics is not self-evident for many people. Talk about 'intrinsic motivation' and of mathematics as 'an end in itself' can sound rather like special pleading: the kind of rationalisation one might expect from any special interest group.[12]

Second, the main sticking point for many people is that they cannot really see the *purpose* of mathematics. The purpose might obviously be sought in applications, but there are relatively few of these, in relation to the total body of mathematics, and the quality of many of the so-called 'applications' found in

textbooks is very poor. These so-called 'applications' often seem to consist in painting a layer of mathematical mystification and symbolism over an otherwise clear and understandable practical situation.

'What is the point of applying maths to a practical situation if the main result is to inhibit one's capacity to think about, and to understand, that situation?' This is the essence of the difficulty many feel.

What is needed to get over the purpose of mathematics to many people is a demonstration set of instances of what may be called *applications with bite*. We mean by such applications the use of mathematical models to *predict* the outcome of developments in situations which are of interest to people *independently* of mathematics.[13] Usually the 'developments' we are concerned with are human inputs into a situation, but in some instances, e.g. predicting the weather, or river levels at Brisbane, the 'development' is simply the product of natural forces.

The basic form of such applications may be summed up in the schema:

$$Su + D = C$$

where Su is an unsatisfactory situation, D is a development of that situation and C stands for the consequences of introducing D into Su. Sometimes we know what C we want to produce and the question is really what D will be necessary (if any) to do this.

D may take the form of *doing* or *building* or, in the inputless cases, simply in the welter of natural forces acting on the initial situation.

In effect we use mathematics to mimic SU and the introduction of D, and our 'modelling' consists in combining an SU with a D to give a mathematical result, C, which we then interpret in terms of the real situation.

We now give four examples of problems of this type.[14]

(1) George is a very poor man, and his only companion is a dog called ROVER. After paying all his other expenses George has only six dollars, forty-nine cents left at the end of a day with which to feed himself and ROVER. For *good nutrition* George needs 2 000 calories a day and ROVER needs 400. George can buy a kind of dog food which will provide ROVER with 4 calories per cent spent on it. He can also buy food for himself which provides 2 calories per cent spent on it. Now George would like to play fair with the dog and provide ROVER with the same fraction of good nutrition as himself. Can he do this? If so, how much must he spend each day on dog food for ROVER?

COMMENTS: The 'unsatisfactory situation', Su, is the poor man and his dog. The desired consequence, C, is that the man and the dog should have the same fraction of good nutrition. We use an equation to find D, that is, the amount George should spend on dog food if he wishes to achieve his declared aim.

(2) Miss Smitherton would not tell reporters her age. 'No actress would willingly reveal her age', she said.
'Shall we say you are twenty-five?' she was asked.
The reporters were stuck. It was well known that Mrs Smitherton (her mother) was twenty-five when the actress was born, but this did not really help, because no-one know Mrs Smitherton's age either. Eventually Miss

Smitherton's manager stepped in and said that the star would answer *one* question truthfully, but would not answer any direct question about her age.
The reporters got together and discussed three questions they thought they might ask Miss Smitherton:
(a) If we halve your age, and halve your mother's age, and subtract the smaller from the larger, what will the answer be?
(b) How many years ago were you (then) half your mother's age (then)?
(c) If we double your age and double your mother's age and subtract the smaller from the larger, what will the answer be?
Which question should the reporters ask?
If Miss Smitherton replies that the answer is 23, what is Miss Smitherton's age today?'

COMMENTS: In this question the unsatisfactory situation is that the actress will not reveal her age plus the information that her mother's age is $x + 25$, where x years is the actress's age. D is the question the reporters should ask in order to be able to deduce Miss Smitherton's age. C is the deduction of Miss Smitherton's age. The question is, what D, when added to Su, will give C.

(3) Jack wanted to ring his friend in Melbourne, but could not remember his friend's telephone number.
'Why don't you call Directory Enquiries and find out?' everyone said.
'The trouble is, I don't know his address in Melbourne, and I don't know his surname either. We always call him "Chuck".'
'Oh! That won't help much!' everyone agreed.
'Well, can you remember *anything* about this number?' Jack was asked.
'Oh, yes . . . it has three digits which increase in steps of one each time; then it has the same three digits again,' said Jack.
'But can't you remember even one digit?' he was asked.
'No, it's gone completely,' said Jack.
One of the group suddenly had a bright idea.
'Look, when you phoned Chuck last time we were using the tape recorder. We might be able to time how long you spent dialling from the recording.'
So they played the tape and found that Jack spent fifteen seconds dialling, of which five seconds was taken dialling the Melbourne code.
'That means that ten seconds were spent dialling the actual number', said the last speaker. 'And that means that the digits must add up to about 30', he added.
Was this information enough to work out Chuck's number? If so, what was Chuck's number?

COMMENTS: Here Su is the unsatisfactory situation in which Jack finds himself, but including the bit he remembers about the number, namely that it consists of three digits x, $x + 1$, $x + 2$ repeated. D is the idea of timing Jack from the old tape recording. C is the reconstruction of Chuck's number from the two pieces of information.

(4) Jill's friends all wanted to know how much her new doll cost.
'I don't know,' she said.
'Well, I overheard your father complaining,' said Ann, her friend.

'He said that he bought this doll through the post and a similar one for your sister and there was a charge of a dollar for posting the two. Then he discovered a shop where you could get two dollars off the usual price if you bought three of these dolls at once.'
'So what?' said Jill.
'Well, the cost of three dolls in the shop was exactly the same as he had paid by buying them through the post,' said Ann.
'How does this information help?' questioned Jill.
Can you work out the price of the dolls from what Ann said? If so, what was it?

COMMENTS: Here the unsatisfactory situation is that Jill's father thought he had made a poor bargain and consequently he would not tell Jill what the price of the doll was. (This was really the 'unsatisfactory' bit as far as Jill and her friends were concerned. They wanted to predict the price of the doll.) D was the discovery that the sale offer in the shop enabled one to get three dolls for the price of two purchased through the mail. C was the price of the doll.

The Analysis of Viability

It hardly needs to be said that problems consisting of 'applications with bite' in the manner of (1) to (4) above can be generated in unlimited numbers. A genre of such problems exists. Opinions may differ on just how 'difficult' they are, but at least they are clearly not simply handle-turning exercises. They lack the chief characteristic of handle-turning exercises; namely, that a child can do a handle-turning exercise without understanding what he or she is doing. So there is really no doubt that questions (1) to (4) would be classified as 'problems' by most commentators.

Nor does it require much insight to see that such problems are essentially straightforward in character. There is a body of linguistic information to be sorted out by the solver but, on being attended to, it *does* fall into place fairly neatly. No-one who has any real experience of solving practical problems using mathematics would find such problems as (1) to (4) difficult. On the other hand, some teachers complain that they find them very difficult and that their classes find them virtually impossible to do.

It appears, therefore, that there is a complete lack of consensus on whether such problems as (1) to (4) are viable or not. When experienced teachers say that these problems are *not* viable, one is bound to take notice: but the explanation may lie in different assumptions about the nature of mathematics education. To make an obvious point: many children acquire an experience of mathematics which is almost wholly based on handle-turning work. It would be surprising if such children did *not* find problems like those listed above difficult. From an early age they have been conditioned to think of mathematics as a kind of ritual, involving writing marks on paper, verbalisations and sometimes manipulating concrete analogues of the written marks. They have rarely, or even never, been expected to *relate* this ritual activity to transactions in the real world. They have simply no expectation of the ritual making sense in relation to real things or machine concerns. What compounds the problem considerably is a corollary of this fact; namely, that some *teachers* are in virtually the same situation. They, too, have no expectation that mathematics can be applied 'with bite', and the whole idea of reconceptualising mathematics as a purposive activity is quite a daunting one for them. But it is only when such teachers actually succeed in

reconceptualising their mathematics as a purpose activity, that they will be able to see the viability of the kind of problems advocated here. In other words, the extreme sense of difficulty which some teachers feel in relation to problems like (1) to (4) above is a product of their reliance on an habitual conceptual framework which virtually *excludes* activity of the kind needed to make sense of (1) to (4).

Given a background of regular applicative mathematics, there is nothing especially 'difficult' or 'formidable' about these problems; in the absence of an applicative background they may, of course, look quite worryingly difficult. At this point some teachers will exclaim, 'Yes, but surely one has to teach them some mathematics before they can ever apply it?'. This is the pedagogic axiom on which so much mathematics education has been founded, and it may seem outlandish to attack it: but the point is that the axiom is mistaken. The fallacy in it had already been exposed by Whitehead in his famous paper to the ICMI in 1912. The axiom presupposes that the essence of mathematics is that it is a symbolic-manipulative ritual which is chiefly an end in itself but which also happens to have a few useful spin-offs. The presupposition can now be seen, in the light of recent thinking on the applicability of mathematics, to be as mistaken as the pedagogic axiom. Any bit of mathematics can serve as a model of some real situation: the whole of mathematics is potentially directly applicable, or at least supportive of directly applicable, mathematics.[15]

No-one would seriously advocate teaching children walking movements before teaching them to 'apply' these walking movements to vertical locomotion! Walking is best learnt by trying to walk, not by learning walking as an abstract idea first, and then implementing it. Mathematics with bite is similarly best learnt by using mathematics from the earliest stages as a means to *predict* the outcomes of human interventions in situations – interventions which are of interest independently of the mathematics.

Some teachers may reject any attempt to rethink mathematics as a purposive enterprise on these lines. The human mind has an amazing capacity to construct rhetorics apparently to justify many such lazy-minded choices. It can be held that applications of mathematics 'sully' the subject, or even 'prostitute' it to materialistic ends! Relative to such a position, problems like (1) to (4) above will not look like, or seem to count as, 'educationally viable' problems. What is being maintained here is only that problems like (1) to (4) above are 'educationally viable' *relative* to an up-to-date general view of the applicability of mathematics, and of the role of this applicability as the main motivator of early mathematical study.

How, then, does the new general view of the applicability of mathematics make a difference? The answer is that it leads us to foster three new, hitherto neglected types of skill:

(a) *the translation skills.* These are skills involved in translating information from words and diagrams into mathematical form and vice versa.
(b) *envisaging skills.* These are the skills involved in filling out the outlines of the situation and seeing it as being 'real'.
(c) *purpose recognition.* Looking for and identifying the most probable purpose or purposive intention underlying a problem.

It is these skills that a regime of problems based on applications with bite helps us to build up. In fact one might say that once a child's applicative skills (a), (b), (c) have sufficiently built up, questions like (1) to (4) above will *ipso facto* be experienced as 'problems'. This is not just a theory. At least in the case of

students in the age range 14-18, a mass of evidence has been obtained that students following an applicative course *do* develop their applicative skills, and *do* subsequently achieve a standard of problem solving in relation to problems of the type illustrated above which exceeds that of more able students following formal, handle-turning courses.[16] If this can be achieved in the age range 14-18 *following* years of prior work of a different kind, how much more might be expected if the 'conditioning' were of an applicative type from the beginning?

The essence of the problem of introducing problem solving work was that of establishing a form of quality control, that is, of ensuring that the problems were going to be 'educationally viable' problems. Investigation of the matter now shows that the bottleneck does not lie so much in testing putative problems for educational viability, as of ensuring a supply of suitable problems. Given a supply of broadly suitable problems, there is no difficulty in arranging pre-testing and other screening devices such as readability and visual intelligibility tests. The problem, therefore, reduces in essence to understanding the idea of applications with bite. This idea, once understood, enables us to generate the problems we need in sufficient numbers that, even after careful screening, there need never be a shortage of suitable problem material.

It is evident, of course, that research on the educational viability of real life applications with bite is only in its infancy. We need to investigate the many varieties of such problems, the general factors making for lucidity of formulation, the relative efficiency of different problems in terms of the amount of mathematical involvement per Ogden of interpretive effort on the part of the child.[17] In this paper it has only been possible to broach some of the initial issues involved.

Real life 'applications with bite' are not only kinds of problems that are needed in schools[18], but they are a category of major importance, because they represent the main channel by which the child can be taught to think about the real world and hence to see the great depth of the relevance of mathematics.

Notes

(1) See Whitehead, A.N., *The Aims of Education*, Benn, London 1929. The lead paper in this collection is Whitehead's address to the Education Section of the ICMI in 1912.

(2) It is evident that the micro-chip is beginning to throw the onus of industrial effort onto *thinking*, rather than merely the repetition of operations. In effect the chip is taking over the repetitive operations.

(3) In other words, there is a vast mass of 'problems' of every shape, size, and degree of difficulty. We need to identify a subset of whose members are intelligible to the average child and which may be administered with beneficial results in the classroom.

(4) *Either* to the teacher *or* the class. A problem which merely *looks* too difficult will be experienced as *being* too difficult.

(5) See Ormell, C.P. 'A Survey of Interdisciplinary Applications of Mathematics: General Design and Development as a source of Selective, Simplified Modelling', *Zentralblatt fur Didaktik der Mathematik*, (1980a).

(6) See Taylor, Ann in *Introducing Psychology* (Ed. D.S. Wright and A. Taylor), Penguin, London, Chapter 16, p.420 (1970).

(7) cf Green, Judith in *Thinking and Language*, Methuen, London (1975): 'A child who has no previous knowledge of long division will obviously be in a totally different position from an adult for whom it is a trivial problem' (p.18). (In the same section Ms Green also offers a general classification of problems into six levels of difficulty.)

(8) See Holt, John, *How Children Fail*, Penguin, London (1969), p.37: 'For children the central business of school is not learning, whatever this vague word means, it is getting these daily tasks done, or at least out of the way, with a minimum of effort and unpleasantness.'

(9) See Ormell, C.P. 'Why Teach Mathematics?', *Educational Magazine*, 35, 5, (1978); 'Values in Mathematics', *Values and Evaluation in Education*, (Eds Straughan and Wrigley), Harper and Row (1980b); *The Applicability of Mathematics*, Maths Applicable Group, Reading (1981).

(10) See *Mathematics Changes Gear*, Heinemann-Schools Council, London, 1975, p.29.

(11) See Peirce, C.S. 'The Essence of Mathematics', *The World of Mathematics*, XII, 1773-1783, Simon and Schuster, New York, 1956.

(12) See Ormell (1980b), p. 227.
(13) A fuller account may be found in Ormell (1981).
(14) Problems (1) and (2) were devised at an MAG Development Workshop in Reading in 1979. A full report may be found in *Modelling with Simple Algebra*, Maths Applicable Group, Reading, 1981.
(15) See Ormell (1981).
(16) See *Sixth Form Maths Project*, Maths Applicable Group, Reading (1979), pp. 19-20.
(17) 'Ogden', a notional measure of intellectual work, coined by Dr Gattegno: see *Mathematics Teaching*, 59, pp.34-42, (1972).
(18) Clearly problem solving within formal mathematics is important for any boy or girl aiming for a career with a theoretical science or mathematics basis.

Modelling and Applications in the Problem Solving Curriculum: Some Practical Problems

Vern Treilibs & John Gaffney

One could be forgiven for thinking that the main aim of mathematics teaching is to recycle mathematics. Little, if any, of the mathematics taught in secondary schools is used by students in solving problems. Most of the skills being learned are *potentially* useful rather than *actually* useful: the mathematical diet lacks an essential ingredient.

Problem solving can be defined as *using mathematics in unfamiliar situations*. These situations may lie within mathematics or they may be from the 'real world'. In either case problem solving is not algorithmic in nature; rather it requires students to achieve a considered synthesis from their mathematical knowledge and skills.

Problem solving has a wide range of meanings. It is useful to distinguish two major areas of problem solving: *abstract problem solving* and *modelling*.

1. *Abstract problem solving* is undertaken within mathematics, although the problems are often set in a pseudo-realistic context. The problems are generally well-defined, and a vast number of them can be found in the literature. They range in difficulty from those suitable for very young pre-schoolers to those set for Mathematical Olympiad contestants. Abstract problem solving is often described as 'thinking mathematically'.

2. *Modelling*, by contrast, always begins in a real, or realistic, situation. Modelling requires extensive use of a range of translation skills before the mathematical model becomes usable. Modelling problems are generally not well-defined and are difficult to find for use by students. Modelling is sometimes seen as 'applications', but great care must be taken not to confuse the process with the standard models presented in past and present applied mathematics courses.

The distinction between abstract problem solving and modelling is, to be fair, not always as clear cut as the above descriptions may suggest. It is certainly possible to model mathematics with mathematics. The interpretation and solution of the models formulated in modelling real situations often require adept problem solving.

In this paper, we propose that considerably more attention and effort should be directed towards the modelling component of problem solving, if necessary

at the expense of instruction targetted on the abstract problem solving component. Concerns that we have about the teaching of abstract problem solving include these:

- despite much research, little is known about effective ways to teach problem solving processes;
- problem solving has low motivational value for many students;
- there are few rewards for partial progress;
- less able students often see the solutions as 'magic wand' approaches.

Some of the points about teaching modelling which attract us include:

- the potentially high motivation;
- the admission of 'correct' solutions of varying quality;
- the same problem can be accessible to students with markedly different mathematical 'tool kits';
- it is patently useful, in that it deals with real problems;
- the modelling *process* accommodates regular additions to the mathematical 'tool kit' of the student, and allows problems to be re-visited.

Whichever component of problem solving is emphasised, students will remain poor problem solvers until problem solving is regarded as more important than the acquisition of 'basic skills'. We are proposing that of the time allocated for the learning of mathematics, about 20% should be deliberately invested in learning and using modelling.

Learning modelling requires the development of an extensive range of skills, in particular technical, tactical and strategic skills. Of these, the last two are extremely complex. The tactical skills include elements of heuristics, whilst strategic skills include the ability to generate variables and to formulate models. For most students, these skills need time to develop but, once acquired, they are widely applicable.

Modelling problems can be drawn from any facet of life which interests the students and lends itself to mathematical analysis. Social sciences and technology are often 'nearer' to more students than the physical or natural sciences, and would be preferred as a source of tractable problems. This is despite the fact that the physical sciences offer a wealth of excellent models. There is a grave danger that the teachers will obscure the learning of the *process* of modelling with learning the *model* instead.

The problems presented can range from *completely open* to *tightly structured*. For example, compare 'The School Canteen has money troubles' with 'Will installing a "Space Invaders" game in the Canteen substantially raise its profits?'. One statement requires students to define the problem, whereas the other provides a relatively tight specification of what needs to be known.

The choice of problems or modelling situations requires careful matching with the following levels of development of the student:

- level of mathematical ability;
- level of experience in the real problem situation;
- level of modelling ability.

Our limited experience in this area persuades us that it is most unwise to expect students who are learning modelling to be expected to apply the most recently acquired content knowledge and algorithms as part of the task. It is unfair, and a total misreading of the inherent difficulties in learning and teaching modelling, to judge the quality of the learning by examining the level of mathematical content used to describe the model. Our initial work leads us to suspect that arithmetic, leading to generalised arithmetic (algebra), and some simple graphs will be the tools most students will use for much of their modelling.

There is an urgent need for a stockpile or bank of modelling problems suitable for secondary students to use. A start has been made on developing such banks (e.g. at the Shell Centre at Nottingham), but their growth has been slow.

There are four major constraints on the teaching of modelling which must be loosened:

1. Mathematics teachers, traditionally conservative, find it difficult enough to accommodate changes in course content. Teaching modelling will almost certainly require some changes in teaching style. This will prove difficult, and may well be impossible for many teachers.
2. Mathematics courses are still commonly defined in terms of content rather than process. Many courses are so loaded with facts to be learned and algorithms to be performed with Pavlovian reflex that there is insufficient flexibility to allow for the kind of experimental work we propose.
3. While assessment styles have certainly opened up from the standard test paper approach, methods which reward individual project work and group effort remain 'poor relations' in many secondary schools. Devising fair 'mark schemes' for mathematical modelling, particularly by group work with teacher consultation, is still in the formative stage in secondary schools.
4. We are short of experience and resources.

Of the possible approaches to reduce the effects of the above constraints, one which seems quite unfeasible would be to mount a major national project. Quite salutary lessons can be learnt from the problems faced by primary teachers during the implementation phase of the USMES project in the USA. In the current climate of declining enrolments and reduced funding for education, it would seem to be most unwise to mount a large-scale initiative. It appears more prudent to embark upon a long-term infiltration of the mathematics curriculum.

Some actions which are feasible and could contribute to a gradual infusion of modelling into mathematics courses, are:

- to write materials on applications to be added (or integrated) to typical chapters of popular texts used with year 7-11 students. While these would certainly not be 'modelling', they would deal with real world situations and thus widen the horizons of teachers and students. Many such problems have 'loose ends'; following them up often involves some mathematical modelling.
- to establish a few 'lighthouse' schools to experiment with the teaching and learning of modelling. It would be necessary to use observation and recording methods which would not only permit reasonable evaluation but provide information suitable for teacher development on a wider scene.
- to promote modelling through local and national publications likely to be read by teachers.

Constraint 2 is difficult to reduce. It probably means that developments should be initiated in junior and non-examination senior courses.

Constraint 3 will only be reduced when enough teachers accept that a major criterion of success is what students are able to do with their mathematics tool kit. This means that achievement needs to be assessed by methods which do not unduly emphasise recall and standard application of algorithms at the expense of higher order processes. We do not suggest that recall and algorithms are unimportant. They are essential components of the problem solver's tool kit, but they are not ends in themselves.

Constraint 4 will only be resolved when people work in the area.

In summary, we suggest that the essential purpose of learning and teaching mathematics is for students to see and feel the power of mathematics as they use it to formulate and improve solutions to their problems. For the majority of students, most of these problems should be real world problems. Mathematical modelling is a skill which all students need to learn.

Another View on the 'Problem solving' Approach to Teaching Mathematics

A. Edwards

Introduction

In this position paper, I wish to outline where I think the efforts of mathematics curriculum planners should be at present. The comments herein should be taken to apply only to that component of the curriculum that is required for most secondary school students. The comments do not necessarily apply to that component intended for students who will use mathematics in their future careers. This paper is concerned with the 'average mathematics student' who undertakes about ten years of school mathematics and who eventually becomes an 'average adult'.

It is assumed that the life of the 'average adult' can be improved by improving his or her mathematical skills. The rest of this paper argues the case for an approach to the school mathematics curriculum which aims to do so and ends with recommendations to curriculum planners.

Why Nothing Changes

Two themes have reappeared in recent times in the debate over what should go on in mathematics classrooms. First, much has been said about the need to return to 'basics'. Generally the protagonists of such a move are hazy on what these 'basics' are. Some employers want facility with numbers, others see a knowledge of plane geometry as another essential component, while there are some who want employees who can think. The second theme, which is supported by many teachers and educators, is a 'problem solving' approach where the emphasis in the classroom is on inquiry and investigation rather than imitation and memorisation. These two themes need not be mutually exclusive or exhaustive with respect to the total set of experiences that can justifiably be planned for a mathematics classroom. However, it appears that since both these approaches have been used before over many decades and neither has emerged as the preferred way of teaching mathematics, we need to think quite carefully before supporting either approach.

The following set of possible reasons for the inadequacies of the above approaches is offered:

1. All teachers are not equally facile with all approaches to the teaching of mathematics. Perhaps some teachers, though judged as successful with some methods, are unable to use some approaches at all. This limitation can hardly be thought disastrous if we are not even sure of the merits of alternative approaches. If this reason were valid, it would certainly be one explanation of why research into teaching methods does not seem to produce clear-cut recommendations. A cynic might think that only

5% of the research into teaching methods give a significant result at the 95% level of confidence, particularly if ignorant of the logic of hypothesis testing. Unfortunately, this would be hard to test since the research that is published is possibly a biased sample of all the research undertaken.

2. Teachers undertaking approaches which are new to them may not have been properly trained, or may not have been given the support necessary for them to adapt. Teaching is not a leisurely profession, and if teachers do not achieve some success with a new approach very quickly, they will want to return to 'tried and true' ways.
3. The success of a teaching approach may be dependent on the characteristics of the children, or perhaps of the social setting. There may be psychological or cultural factors which prevent a particular child learning from a particular method. For example, a child or a culture which holds a belief in the absolute authority of a teacher is not going to question that teacher. Do children have to be trained to learn from different approaches?
4. The approaches may be inadequate for some more basic structural reasons (e.g. insufficient time might have been allowed for assimilation of concepts, or there might have been a lack of intrinsic motivation).

If the first is the principal reason for the inadequacy of any particular approach, then the best that can be hoped for is to develop a series of approaches that leads towards whatever aims are set for the mathematics curriculum and then help each teacher discover which approaches suit best, and what the teacher's personal contribution to the aims should be. If the second is true, then we need to provide the appropriate support. But whether it be the first, second or third, we must ensure that as much emphasis is placed on how the curriculum is implemented, as on the actual learning experiences that it comprises. It is futile designing mathematical experiences that teachers with their current resources and skills cannot manage. (How many teachers in open area schools have drawn the curtains across?) If a teacher has an unsuccessful association with the implementation of a different approach to the curriculum then it also may jeopardise the success of future attempts at change.

Where Are We Going?

The aims of the mathematics curriculum may be worthy but, since a child is expected to spend at least ten years of school learning mathematics, we should expect some lasting and positive results from this large investment of time and effort. Why do we find, as reported in a recent TV news items, that university students who have just left a high school were not able to add two single digit numbers? Perhaps, we could argue, it is not necessary to be able to do simple mental arithmetic in these days of calculators, but we should still wonder at why someone who spent at least ten of the last twelve years working with numbers cannot perform such a simple task.

If we were to argue that 'the development of problem solving skills' is a more worthy aim than the ability to do mental arithmetic, could we expect that, a few years after completing ten years of school mathematics based on a 'problem solving' approach, our future young adults will retain a large proportion of these skills? Certainly whatever our aims, if we cannot guarantee reasonable retention, we cannot justify the large investment involved. This suggests that we think carefully about the kind of mathematical skills an adult might need and think also about how we can ensure that the skill is there when she needs it.

Mathematical Skills Needed By An Adult

The mathematical skills (using the word in its broadest sense) needed by the average adult not engaged in technical employment should be the subject of a continuing debate in mathematics education circles. The Open University has established a post-graduate course for teachers, Mathematics Across the Curriculum PME 233, in which the school mathematics curriculum is constantly being compared with the needs of adults. In this course the rationale for the need for mathematics is said to be the power that mathematics has to explain and represent everyday phenomena. The types of skills the course emphasises are as follows: planning; measuring; seeking out relationships; making sense of space; making decisions; and representing for understanding. This is the same kind of inventory that one might find as the goals of any 'problem solving' approach to the curriculum. However, the validity of these goals would depend very much on how they were interpreted in the experience of learners.

Before continuing with the needs of adults, we should clarify the use of the terms being used. The term most in need of clarification is the term 'problem' itself. It has been used in the past to indicate anything from an exercise involving the application of an algorithm through to a wild goose chase. Perhaps it is too subjective in its meaning to have any value as a descriptor at all. It is often used to suggest that 'this question is something we can get our teeth into', when many students can despatch it immediately, and others cannot see any point in the question at all. Perhaps the only person with the right to nominate a question as a 'problem' is the person who decides to tackle it. On his recent trip to Australia, Christopher Ormell suggested that, rather than teach 'problem solving', we should teach children to 'think mathematically'. This assumes that mathematics has a unique contribution to make in the job of teaching children and adults to think. Let us examine what this contribution could be and drop the term 'problem solving' from the language of mathematics education from now on.

What do adults do that requires mathematical thinking? Three broad areas of adult activity come to mind. First, they want to make sense of themselves and their environment. At an elementary level they may want to explain why their car ran out of fuel; at a higher level it may be a desire to understand inflation; in general, they are looking for patterns and relationships. Secondly, they make plans, from budgeting to ways of allocating their time. Thirdly, they take decisions, about such things as purchases, jobs and taking chances.

Many topics in mathematics can contribute to these adult needs. However, these mathematical tools should certainly be considered useful: number combinations, measurement, ratio and proportion, scale drawing, tabulation of data, graphs, probability, statistics, flow charts, elementary geometric concepts and skills. Such topics are the subject of the Open University PME 233 course, Mathematics Across the Curriculum, along with the Nuffield Mathematics Project and other projects. Therefore, their details will not be discussed here, but rather how they should be applied will be considered.

What Should Children Do?

Earlier it was stated that we should ensure that mathematical skills are retained into adulthood and are available when needed. Their availability will be related to how well some situations cue appropriate mathematical ideas and how effectively these ideas have been retained.

If a person's car breaks down, he has a problem. He may have to make a decision whether to buy a new car or repair the old. Thinking mathematically, he plans a procedure which will help him make his decision on economic grounds. He estimates the cost of repairs, the expected life of the car following the repair, and compares this with the extra cost of a new car in terms of depreciation, etc. over the same period. Perhaps he acted in this way because when he was at school he was expected to make a similar decision when he buckled the wheel on his bicycle. Maybe this scenario is too contrived. But this is the sort of link between adult needs and school mathematical experience I think we should aim for.

Actually the bicycle could be a very fruitful source of activities since an understanding of the relationships involved in gears, and in braking distances with respect to speed can be applied directly to motor vehicles. Let us consider what the characteristics of these school activities would be that enable our future adult to think mathematically.

1. The activity should involve a question to be answered. The answer may be in the form of a relationship, a plan or a decision. We are preparing children to take responsibility for their own lives.
2. The activity should be immediately practical. One should be able to apply any results in a practical way. To improve the chances of a student finding a worthwhile problem, the scenario of the activity should be familiar and ideally be comparable to an equivalent situation in the adult world.
3. The reasonableness of any results should be able to be verified in a practical manner. This should increase the confidence of students in mathematics. Many students are amazed that the formal predictions of mathematics agree closely with reality.
4. The requisite mathematical skills should be familiar to the students. Unfamiliar concepts from mathematics or other fields should be avoided. It is very easy to frighten students with no background in physics by using concepts such as energy in the specification of an activity.
5. The activities should be designed and managed so that students can experience some creativity. In English lessons, students are allowed to write their own compositions from an early age, in art they produce their own masterpieces but in mathematics, rarely is a student able to claim that a piece of work is his own work. Students spend virtually all of the time in mathematics imitating their teachers and this does not develop their confidence. We need to provide opportunities for them to put some of themselves into their mathematics, and thereby let them truly master mathematics.

The activities with these five characteristics will not be the whole of the mathematics curriculum; there certainly needs to be some teaching of mathematical skills as well. A balance has to be struck between attaining and using skills. Attaining a skill facilitates using it, while using a skill motivates attaining it. For both of these components of the mathematics curriculum there should be opportunities for activities outside the classroom. It is important that mathematics generally becomes associated with a broader range of physical settings than the inside of a classroom. For example, we could practice construction skills in geometry with string and tent pegs on the school oval.

If children are to be involved in the kind of activities sketched above, teachers will need the resources, both in terms of the ideas and the teaching skills required to manage the activities and the time (within that allocated for mathematics) to program them.

What Can Mathematics Educators Do?

The first step in providing activities, such as suggested above, is to determine the kind of things the average adult should be capable of doing with the help of basic mathematical skills. How could mathematics be used to better understand the environment, to plan or to make decisions? What mathematical skills will be needed? In a general sense, the skills required have been listed in many sources, but there has not been any concerted effort to ensure that these skills will be available when needed, or to ensure that they can be applied in the real situation. Given the low mathematical facility of the general public, perhaps mathematics educators have become despondent and now feel that mathematics has no part to play in the quality of life of the average adult. However, surely such weaknesses in the community as the rip-offs perpetrated by 'loan sharks' and the apparant arbitrariness of government decision making are cause enough for mathematics educators to review the efforts in this area.

The next step for mathematics educators is to translate these 'adult' problems into a context understandable and real to children. Thus the 'braking distance of cars' can become 'the braking distance of bicycles', 'household budgeting' can become 'personal budgeting', and 'the building of free standing structures in the garden' can become 'geometric design of models' which may result in the building of the models in their craft lessons. The links do not have to be as strong as in these examples, but there must be some advantage in having this relationship rather than always being involved either in contrived situations or in remote but real ones.

Some of the activities may have direct application within the classroom itself. For example, before adjusting a set of adjustable book shelves one could establish the distribution of books with respect to height and make the decision on the basis of that information, or, before placing the bookcase within the classroom one could establish how it will be used, collect some appropriate facts and make an informed decision.

Given that we were to develop activities with the five listed characteristics, their packaging becomes the next important step. First, the amount of time allocated to the mathematics curriculum in schools has to be reapportioned to allow for them. It also needs to be decided whether the activities are to be totally integrated with the remainder of the mathematics curriculum or whether they are to be treated separately although, of course, they must be programmed to match a developing list of skills. Since the approach to be used in the activities will be different from the usual mathematics classroom experience, they may well constitute a separate, recognisable component. This will enable a teacher to develop better an alternative teaching style for using the packages. It means that the packages must contain comprehensive guidance for teachers, including a statement of the philosophy behind them, as well as instructional resources for the students.

With respect to a proposed time allocation, I would suggest that the activity packages be allocated 20% of the time allocated to mathematics for school years six to ten, while each package might, on average, be designed to span two weeks (i.e. 20% of the time spread over two weeks). There are several reasons

for such an allocation. Any more time would be virtually impossible to wrench from conventional activities, yet for ideas to develop, time is of the essence. Spreading an activity over two weeks will allow more time to think about strategies or collect factual information.

As stated earlier, the manner of implementation of a curriculum is as important as the curriculum itself. While there may be examples of packages for the mathematics classroom which do focus on real problems, there has not been a concerted effort to put as much effort into implementation as there ought. Some new developments require the teacher to be wholly converted to a new approach. I think this is unrealistic. The teacher needs some firm, familiar ground on which to stand between bouts of trying her wings with new methods.

Conclusion

I would like to emphasise that the suggestions are a response to the need for a curriculum which improves mathematical thinking. The 'average' adult will not want or need to solve mathematical 'problems' but he may apply his mathematical skills to 'everyday' problems. Let us put our efforts where the need is. Although the suggestions have been made with the 'average' adult in mind, there is a possibility that the student bound for a career in applied mathematics might be better served as well. In the past we have tried teaching so-called problem solving skills directly, but this seems to encourage an algorithmic approach. In this paper I have suggested that we focus on improving mathematical thinking, and that the pedagogical approach used should be based on establishing motivation and confidence rather than on any heuristic techniques.

Summary of the Problem Solving Inventory

ORGANISERS	PROCEDURES	SKILLS
01.1 What does the problem tell me? 01.2 What does the problem ask me? 01.3 What can I introduce to help get started?	P1.1 Explore the problem P2.2 Make and test guesses P1.3 Define terms and relationships P1.4 Extract information P1.5 Organise the information P1.6 Introduce a represention P1.7 Introduce a form of recording	**For Handling Information** S1.1 Indentifying information S1.2 Collecting information S1.3 Recording information S1.4 Sorting and ordering S1.5 Presenting information
02.1 Can I make connections? 02.2 Is there a result which will help? 02.3 Is there a pattern? 02.4 Can I discover how or why? 02.5 Can I break down the problem? 02.6 Can I change my view of the problem?	P2.0 Be systematic P2.1 Search for relationships P2.2 Analyse relationships P2.3 Make simplifying assumptions P2.4 Find properties the answer will have P2.5 Try particular cases P2.6 Adjust guesses P2.7 Formulate & test hypotheses P2.8 Try related problems P2.9 Control variables systematically P2.10 Use one solution to find others P2.11 Work backwards P2.12 Focus on one aspect of the problem P2.13 Eliminate paths P2.14 Partition the problem into cases P2.15 Reformulate the problem P2.16 Upset set P2.17 Develop the recording system P2.18 Change the representation P2.19 Make a generalisation	**For Representing a Problem** S2.1 Choosing a mode S2.2 Using a representation S2.3 Translating between representations **For Enumerating** S3.1 Scanning all possibilities S3.2 Eliminating needless repetition **For Finding Patterns** S4.1 Recognising patterns S4.2 Predicting from a pattern
03.1 Is the resolution acceptable? 03.2 What can I learn from the resolution? 03.3 Can I extend the resolution?	P3.1 Check P3.2 Look back P3.3 Communicate P3.4 Find isomorphic problems P3.5 Extend to a class of problems P3.6 Create different problems	**For Testing** S5.1 Testing a result S5.2 Testing a hypothesis S5.3 Testing an argument

CHILD'S INPUT

MATHEMATICS REQUIRED
Drawing ability, the lines must be long enough to enable new lines to cut them.

OTHER INPUT

TEACHER INPUT

Encourange children to draw up a table and look for a number pattern.

Can each new street cut all the existing ones?

THE PROBLEM
In a village there are three streets. All the streets are straight.

One lamp post is put up at each crossroads.

What is the greatest number of lamp posts that could be needed.

Now try four streets. What about five streets?
Try six streets.

Ruler, pencil, paper.

A geoboard may be helpful for younger children.

OUTPUT

ENTRY 1.1, 1.2
ATTACK 2.10, 2.11
EXTENSION 3.3
SKILLS
REPRESENTATION 4.2
INFORMATION ANALYSIS 5.3

MATERIAL INPUT

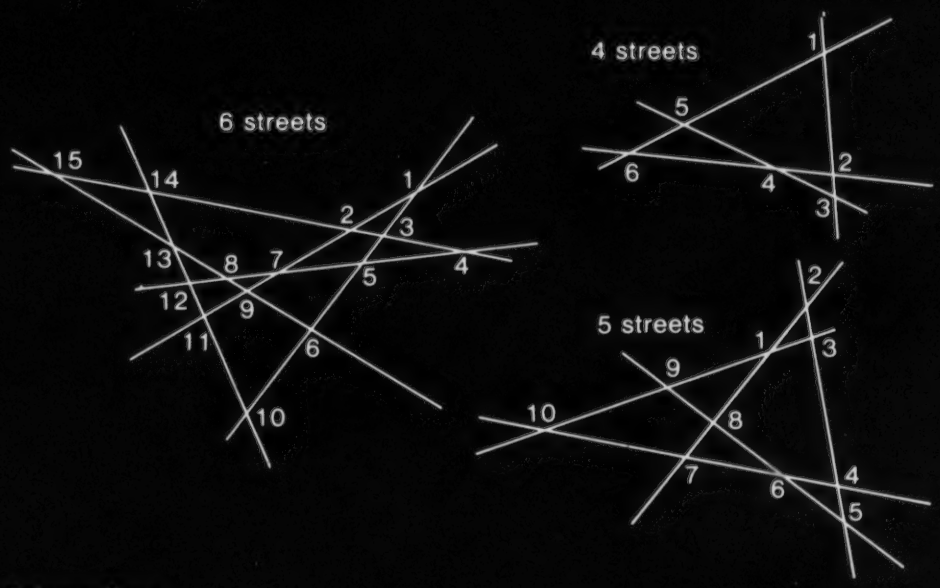
OPERATIONAL LEVEL Concrete

Resolution and notes

There is a tendency to draw the streets parallel to or at right angles to each other. This 'set' does not give the greatest number of crossroads. (2.11) Each new road added can be drawn to cross all the existing roads. Exploring, 1.1, or playing with the problem is usually involved in the Entry process.

Number of sts.	Number of posts.
1	0
2	1
3	3
4	6
5	10
6	15

The number pattern involved is that of the triangular numbers, numbers which when presented by dots can be arranged as triangles.



Interventions

A single line to represent each street is better than two parallel lines, with the latter the diagram is harder to draw for larger numbers of streets.

Extensions

Triangular numbers occur in quite a number of contexts and problems.

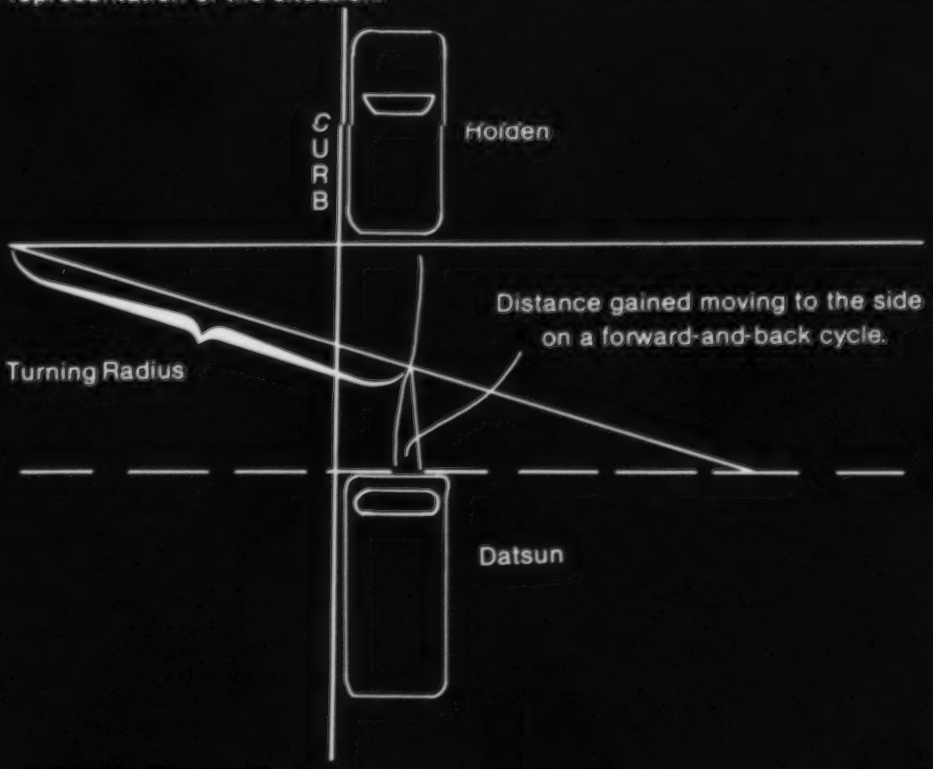
Summary of the Modelling Inventory

ORGANISERS	PROCEDURES	SKILLS
01.1 What does the situation communicate? 01.2 Does the situation suggest a problem? 01.3 Do I need additional information?	P1.1 Explore the problem P1.2 Make and test guesses P1.3 Define terms and relationships P1.4 Extract information P1.5 Organise the information P1.6 Introduce a represention P1.7 Introduce a form of recording	**For Handling Information** S1.1 Indentifying information S1.2 Collecting information S1.3 Recording information S1.4 Sorting and ordering S1.5 Presenting information
02.1 Can I symbol the information? 02.2 What are the variables? 02.3 Is there a function or relation in the situation? 02.4 Can I graph the situation? 02.5 Can I represent the situation by drawing, diagramming or writing a model?	P2.0 Be systematic P2.1 Search for relationships P2.2 Analyse relationships P2.3 Make simplifying assumptions P2.4 Find properties the answer will have P2.5 Try particular cases P2.6 Adjust guesses P2.7 Formulate & test hypotheses P2.8 Try related problems P2.9 Control variables systematically P2.10 Use one solution to find others P2.11 Work backwards P2.12 Focus on one aspect of the problem P2.13 Eliminate paths P2.14 Partition the problem into cases P2.15 Reformulate the problem P2.16 Upset set P2.17 Develop the recording system P2.18 Change the representation P2.19 Make a generalisation	**For Representing a Problem** S2.1 Choosing a mode S2.2 Using a representation S2.3 Translating between representations
03.1 Is the model acceptable? 03.2 Does the model predict? 03.3 Is the model partial or total in explanatory power? 03.4 Can the model be extended to account for more phenomena? 03.5 Does the model suggest other questions?	P3.1 Check P3.2 Look back P3.3 Communicate P3.4 Find isomorphic situations	**For Enumerating** S3.1 Scanning all possibilities S3.2 Eliminating needless representatives **For Finding Patterns** S4.1 Recognising patterns S4.2 Predicting from a pattern **For Testing** S5.1 Testing a result S5.2 Testing a hypothesis S5.3 Testing an argument

Modelling Situation

Resolution and Notes

Students may need help in realising that successive sectors of circles with centre points 'walking out' on two lines tangent to the bumpers is a reasonable representation of the situation.



Curricular Level – year 11

Interventions: It is sufficient to trace the path of a single point, say the mid-point of the back axle. However, some students will be helped by correlating the angle of the front wheels with the path of this point. Why can the car gain on moving sideways only when pulling forward?

Connections: The model is related to the classic problem of the snail in the 30 metre well that would climb 3 metres each day only to slip back 2 metres while sleeping each night.

Extensions: What happens if the distance between the cars is changed? If you had a car with shorter turning radius?

Modelling Situation

Student Input

Mathematics Required
1. Pythagorean Relation
2. Geometry of Circles

Other
1. Knowledge of the turning radius of cars, width, length, etc.

Teacher Input
1. Request identification of path a car will follow on pulling up-and-back while trying to edge to the side.
2. Is there a curve to draw that is a reasonable representation of this path?
3. Help students use their familiarity with cars to identify useful data.
4. Is the length of the car relevant information?

The situation. Your Toyota Corolla is parked parallel to the curb. You return to find it squeezed between a Holden and a Datsun van with only .6 metre free space between the bumpers. You don't want to wait for the other drivers!

Output

1. Compass, straight edge, pencil and paper.
2. Scientific calculator.
3. Specifications of cars (optional).

Materials Input

The Role of Problem Solving in School Mathematics: A Position Paper of the Australian Mathematics Education Program

Marjorie Carss and Alan Osborne

Introduction

The Australian Mathematics Education Program (AMEP) is a Curriculum Development Centre project based on a co-operative model of working with education systems, institutions and teachers in the states and territories. In announcing funding of AMEP, the Curriculum Development Centre issued a media release which indicated that AMEP would address a wide range of problem areas within mathematics education as well as provide some documentation of current practice (i.e. summary of syllabus documents, review of resources used).

The overall aims of AMEP are

- to assist teachers to be more effective by helping to develop their confidence and competence in teaching mathematics,
- to help students develop positive attitudes to learning mathematics, and
- to understand better the meaning, relevance and importance of mathematics in an advanced industrial/technological society such as ours.

While we are cognizant of the criticisms levelled at schools by industry, commerce, tertiary institutions and politicians, we have no evidence that standards in mathematics, as they have been measured, have declined in recent years. We are, however, aware that the expectations of tertiary institutions have not been modified despite the increased range of ability of students entering these institutions. AMEP strongly suspects that the expectations of industry have increased. The application of technological advances and increased mechanisation of routine tasks within industries implies a demand for successful performance at higher cognitive levels than in the past.

It is becoming increasingly important that students not only know their number combinations but be able to apply this knowledge and understand the basic processes involved. This is important not only for arithmetic, but for those other basic mathematical processes (such as principles of measurement, spatial relationships, classification, recognition and development of pattern and structure, etc.) that are not so readily identified by the general public as being central goals of mathematics education.

Thus, the more immediate aims of AMEP are to assist teachers so that they can better help their students

1. attain those commonly shared goals appearing in appropriate syllabus documents,
2. become confident in doing and using mathematics, and
3. be better able to apply and use mathematics in everyday life and in work situations.

It was clear even from the beginning that AMEP could not address all issues raised in the media release at one time. A meeting of representatives of education authorities in all States and Territories met to establish priorities for action within AMEP. AMEP has elected to concentrate on the many aspects of problem solving in mathematics as a major means of helping teachers attain these broad goals. Problem solving in the teaching of mathematics can serve:

1. to improve student understanding of the application and power of mathematics;
2. to facilitate the retention of basic mathematical knowledge and skills;
3. to provide access to a wide range of embodiments of mathematical principles and examples of fundamental mathematical processes; and
4. to demonstrate and confirm the use and appreciation of general problem solving heuristics applicable in mathematics and more generally in all areas of the school curriculum.

Problem solving permeates doing mathematics wherever the new and challenging is encountered, whether in a purely mathematical setting or in a situation demanding the application of mathematics in a particular context. It would be easy to conclude from this statement that mathematics that is taught or developed without attention to problem solving is neither honest nor comprehensive in communicating the nature and character of mathematics and its uses.

What is Problem Solving?

The term 'problem solving' tends to create some confusion when it is used in the context of considering curriculum and instructional issues because of the many settings in which the word 'problem' is used and the different intents in using it. In discussions about the place of problem solving in mathematics education, we note that it is used to communicate about:

- teaching *for* problem solving (focussing on the acquisition of concepts and skills useful to a student for solving problems);
- teaching *about* problem solving (focussing on learning strategies, heuristics, and the processes and behaviours of problem solving); and
- teaching *through* problem solving (focussing on the use of problem solving as a teaching methodology).

In short, confusion arises because of the many different goals, processes, skills and concepts that can be considered and the many different educational functions that can be encompassed under the phrase 'problem solving'.

Problem solving possesses a wide variety of meanings for various people. Among the activities widely associated with problem solving are:

- solving word problems,
- solving non-routine mathematical problems,
- solving application problems,

- solving modelling problems,
- solving logic problems.

Each of these activities in and of itself is too narrowly conceived to constitute problem solving. AMEP believes that problem solving must be broadly enough defined to encompass situations that involve all the mathematics appropriate for a comprehensive curriculum of school mathematics. The following definition of 'problem' meets this criterion:

> A *problem* can be thought of as a situation that poses a question or defines an objective in the light of some given information or conditions where the individual attempting to answer the question or meet the objective does not possess an immediate solution.

The solution process requires an active search, prior knowledge of some mathematics, and a repertoire of techniques or strategies. To teach problem solving means that the teacher must be concerned about the active search (attitudes) and techniques or strategies (processes) as well as skills and knowledge (content).

Some problems are better suited than others in helping teachers establish this trilogy of attitude, process and content. A useful set of criteria for judging the suitability of a problem has been given by Nelson (1980):

1. The problem should have demonstrable significance mathematically.
2. The situation in which the problem occurs should involve real objects or simulations of real objects.
3. The problem should be interesting to the child.
4. The problem should require the child to make some modifications or transformations in the materials used.
5. The problem should allow for different levels of solution.
6. Many physical embodiments should be possible for the same problem.
7. The child should believe the problem can be solved and should indicate when a solution has been reached.

Developed for selecting problems for young children, these criteria are easily adaptable for older students who can also deal with complex, real life situations many of which only have partial solutions.

Problem solving has long represented an important goal for mathematics teachers. In the past, to teach problem solving has been to help learners recognise problem categories each of which is to be 'solved' by the almost algorithmic application of a prescribed rule for solution. For example, a year eight student might well have been taught to recognise that, in a situation describing distance and rate, the relationship usually applied is $d = rt$. If we are to consider problem solving as a basic skill, then we must teach processes that are broadly applicable to a wide variety of situations and contexts. Learners need mathematical power that extends beyond the recognition of problem types.

We argue that learners need experience with two types of problem solving strategies or processes in school mathematics. The first type is global in character, subsuming and structuring the use of the second type of process. Polya's

How To Solve It (1945) provides one outline of a global problem solving process: understand the problem, devise a plan, carry out the plan, and look back. Alternatively, the dynamics suggested by the sequence of words: analyse, design, explore, implement, and verify provide another choice.

These stages in problem solving serve to structure the problem solving act and are effective strategies for many applications of mathematics. The stages can be explored at a depth appropriate to the mathematical maturity of the learner. We believe all learners should have repeated contacts with these or some other general structures that serve to organise the problem solving act.

Processes in the second category are not so broadly applicable and are strategies that fit within one or more of the stages listed above. Many useful descriptions of such strategies can be found. A substantial but still incomplete list of such strategies includes:

- Look for a pattern
- Construct a table
- List all cases
- Act out the problem situation
- Make a model
- Make your best guess
- Work backwards
- Make a drawing or diagram
- Restate the problem
- Write an open sentence
- Solve a more simple but related problem
- Identify what is wanted, what is given and what information is needed
- Specify subgoals

Opportunities to learn these and other particular strategies should be provided for all students at several stages in their schooling so that they can be clearly identified and learnt. The recognition that such processes are important in achieving the goals of the mathematics curriculum but are under-represented in what is usually presented, identifies areas of need for research, development and in-service.

Implications for Curriculum and Instruction Through Applications and Problem Solving

In most mathematics texts and in many typical classrooms, student activity focusses on the development of mathematical knowledge and skills before students are presented with applications or problems that embody those newly acquired skills and knowledge. Applications and solving problems are used as means for practising mathematical skills and processes, and the mathematics to be used in solving the problems presented is usually clearly identified. While this particular role of applications and problems is important in providing a basis for practice, it is the intention of AMEP to develop strategies and materials to extend this role. Problem solving can serve equally well as a strategy for, and a method of, establishing new mathematical concepts and skills. It can also serve as a useful strategy for confirming earlier learning and adding meaning by demonstrating applications in a meaningful context. Problem solving (as an instructional

strategy) is an appropriate context when used to teach both current and new curricula.

A commitment to using problems as a means of learning mathematics and making it meaningful and useful implies that teachers and pupils will each need to feel that problems and problem solving activities are an integral part of the mathematics curriculum and not an extra, joined on to current topics. Some lessons will need to begin with questions about situations in mathematics or the real world, rather than directions on how to follow a pattern or develop an algorithm. Some lessons that aim to teach rules should begin with a situation that leads to the rule as an outcome, rather than begin with a statement of the rule followed by practice in using it. Ventures in problem solving will be judged in terms of their contributions to the student's acquisition of mathematical ideas and of problem solving strategies, to the understanding of methods of applying mathematics and using it as a modelling device, and in terms of their motivational force. Curiosity and a willingness to explore mathematics must be a fundamental attitude to learning current curriculum content.

AMEP is concerned in the long term to shift the context of problem solving in the mathematics curriculum towards applied problem solving. This implies changes in teaching methodology, learning experiences and course content in many classrooms. Solving problems arising in the real world in the mathematics classroom can make a unique contribution to learning about the world. But there is a more fundamental purpose than this. Greater familiarity with applied problem solving should help students move more readily from school into the work force. For students entering tertiary education, the intellectual tools and knowledge of the real world attained through applied problem solving should provide more power in dealing with new learning.

Mathematical modelling is recognised as one of the major ways of bringing a predominantly problem solving focus to work in the applications of mathematics. For more mathematically mature students, situations that require the generation of hypotheses to fit the relationships and facts of the situations, together with testing the model, are necessary for a thorough understanding of the processes of mathematical modelling involved. Because the growth of understanding of such complex processes is seldom an all-or-nothing affair, young learners need experience in applied situations where the mathematical model is not complex. Modelling in the classroom requires that a few problems be thoroughly worked through, rather than that many be touched on superficially. As with instruction on problem solving processes, students need to be encouraged and assisted to talk about the processes and thinking involved as this helps promote psychological closure on the ideas that characterise good learning. Talking about the processes in the context of applied problem solving is, if anything, even more important than where skills, rules and concepts are the objectives of instruction.

Establishing a Problem Solving and Applications Emphasis in Curriculum

AMEP recognises that moving towards an increased emphasis in problem solving and applications, the mathematics curriculum requires several kinds of changes in our curriculum and instructional processes. The steering committee has identified the following areas of need as being particularly important:

(a) To make the mathematics syllabus that each teacher uses oriented more to processes, to reflect strategies and heuristics of thinking as well as the mathematical content of the curriculum.

(b) To adjust and modify teaching strategies and methodology so that they are consistent with an emphasis on problem solving and applications.
Although many teachers use a variety of teaching strategies, a significant number (usually in accord with the texts available to them) depends primarily on a strategy of
(1) state the rule,
(2) demonstrate examples of the use of the rule, and
(3) provide exercises for practice of the rule.
A focus on problem solving and applications demands increased emphasis on
(1) how to develop and present problem situations,
(2) how to focus activity and discussion on heuristic processes, and
(3) how to help students attain psychological closure on the processes.
For many teachers, making such fundamental changes in instructional strategies would be greatly facilitated by the support of in-service education programs and the development of support services.

(c) To extend assessment procedures and testing instruments so that an emphasis on processes is included.
We anticipate that timed pencil and paper instruments will not have sufficient power and flexibility to assess performance in the processes of problem solving and applications of mathematics.

(d) To develop better evidence about which heuristic strategies relate best to the stages of mathematical development and maturity and to design curriculum sequences to match the evidence.
While we currently know about a variety of useful strategies, we do not know enough about their relationship to the learning of mathematics to fit instruction to the stages at which they are most appropriate for children to learn them. Even though we know a wide range of strategies, we tend to feature the same few strategies repeatedly through several year levels rather than make sure that students encounter a large number of different strategies in the course of their schooling.

(e) To make available for teachers, resource materials for problem solving and applications.
Teachers need collections of problems and examples of applications that have clearly identified and verified characteristics in terms of establishing mathematical ideas and the problem solving processes involved. Most exercises in textbooks are oriented primarily in terms of the mathematical principle or rule for which they provide practice. Problem banks need to be catalogued not only by the content skills involved but also the heuristic strategies that should be a product of the learning as well as consider instructional techniques. Since mathematical modelling is relatively new and distinctive for the curriculum, teachers would find problems that exemplify this tool of applied mathematics quite helpful. Another type of helpful resource is a compendium of descriptions of research and evaluation findings for curriculum and instruction.

These areas of need will be satisfied overnight. Some of the needs are, indeed, long-term challenges to the teaching profession that require commitment and attention for several years.

Selected Bibliography on Problem Solving

Selection is based on availability, some evidence that the materials or ideas have been used and that the selection involves some aspect(s) of problem solving identified in the position paper for AMEP.

Burkhardt, Hugh. *The Real World and Mathematics.* Nottingham, England: Blackie & Sons, 1981.
Key concepts and processes for applying mathematics are discussed in terms of classroom practice and their relationship to mathematical skills and processes.

Burton, Leone. *The Skills and Procedures of Mathematical Problem Solving: Report of an S.S.R.C. Sponsored Project at the Polytechnic of the South Bank.* London: Social Sciences Research Council, 1980 (Project Number HR5410/1).
The report describes a useful way of viewing problem solving activities with children and describes problems in terms of teacher inputs, mathematical content, and interventions.

Goldin, G.A. & McClintock, C.E. *Task Variables in Mathematical Problem Solving.* Columbus, Ohio: ERIC/SMEAC, 1979.
A number of variables affecting the teaching and learning of problem solving are carefully described.

Greenes, C., Gregory, J., & Seymour, D. *Successful Problem Solving Techniques.* Palo Alto, California: Creative Publications, 1977.
A practical 'how-to' publication discussing several heuristics. Each problem appears in terms of hints and of complete solutions.

Hatfield, L.L. *Mathematical Problem Solving: Papers from a Research Workshop.* Columbus, Ohio: ERIC/SMEAC, 1978.
Five papers provide a state of the art review of research and research needs relative to problem solving.

Journal of Mathematical Modelling for Teachers. Subscription information: Two issues per year available for US$6.00 from Department of Mathematics, Cranfield Institute of Technology, Cranfield, Bedford, MK43 OAL.
The papers appearing in this journal describe interesting situations that frequently are easily adaptable to the secondary school mathematics classroom.

Krulik, Stephen (ed.) *Problem Solving in School Mathematics, 1980 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics, 1980.
A compendium of papers treating many topics concerned with teaching problem solving. Includes an annotated bibliography.

Krutetskii, V.A. *The Psychology of Mathematical Abilities in School Children.* (J. Kilpatrick & I. Wirzup, eds) Chicago: University of Chicago Press, 1976.
Extensive interviews of students solving problems provides interesting and useful insights on problem solving processes.

Lesh, R. Mierkiewicz, D., & Kantowski, M. (eds) *Applied Mathematical Problem Solving.* Columbus, Ohio: ERIC/SMEAC, 1979.
Issues concerning applying mathematics, interest, motivation, information processing and trends in research are enunciated in this set of nine papers.

Mathematical Association of Western Australia. *100 Mathematical Problems for Students Aged 11 to 13 Years.*
The problems, many classic, along with their solutions, are presented in a form readily usable by teachers.

Meiring, Steven P. *Problem Solving – A Basic Mathematics Goal. 1: Becoming a Better Problem Solver.* Columbus, Ohio: Ohio State Department of Education, 1980.
This in-service document is designed for study by teachers of levels K to 9 and provides examples of a number of heuristic strategies.

Meiring, Steven P. *Problem Solving – A Basic Mathematics Goal. 2: A Resource for Problem Solving.* Columbus, Ohio: Ohio State Department of Education, 1980.
Designed for in-service use, this booklet contains short self-study units concerned with teaching methodology.

Polya, G. *How to Solve it.* (2nd ed.) New York: Doubleday & Company, Inc., 1957.
A classic book in mathematics education that is a first clear exposition of heuristic strategies in problem solving.

Schoenfeld, Alan H. 'Teaching Problem Solving Skills'. *American Mathematical Monthly.* 1981, 87, pp. 794-805.
Based upon a comparison of novice and skill problem solvers solution styles, the role of general problem solving strategies is compared to specific heuristics.

Schools Council Sixth Form Mathematics Project. *Mathematics Applicable Series.*
London: Heinemann Educational Books Ltd. 1978. Titles available.
For teachers: *Teaching Mathematics Applicable.*
For students: *Mathematics Changes Gear; Understanding Indices; Geometry from Coordinates; Introductory Probability; Polynomial Models; Vector Models; Algebra with Applications; Calculus Applicable; Logarithmic/Exponential.*
Mathematics instruction is focussed on applications with a considerable emphasis on modelling.

Wickelgren, W. *How to Solve Problems.* San Francisco: W.H. Freeman, 1974.
An elaboration and extension of Polya-like heuristics is provided in artificial intelligence mode.

The UMAP Journal: The Journal of Undergraduate Mathematics and its Applications. Address correspondence to Ross L. Finney, EDC/UMAP, 55 Chapel Street, Newton, MA 02160, USA Cost US$36 per year for 4 issues.
Articles discuss a wide-ranging set of application problems many of which are understandable to and solvable by 11 and 12 level students.